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Families of $N = 2$ strings

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Abstract: In a given 4d spacetime bakcground, one can often construct not one but a family of distinct $N = 2$ string theories. This is due to the multiple ways an $N = 2$ superconformal algebra can be embedded in a given worldsheet theory. We formulate the principle of obtaining different physical theories by gauging different embeddings of the same symmetry algebra in the same "pre-theory." We then apply it to the $N = 2$ strings and formulate the recipe for finding the associated parameter spaces of gauging. Flat and curved target spaces of both $(4, 0)$ and $(2, 2)$ signatures are considered. We broadly divide the gauging choices into two classes, denoted by α and β, and show them to be related by T-duality. The distinction between them is formulated topologically and hinges on some unique properties of 4d manifolds. We determine what their parameter spaces of gauging are under certain simplicity ansatz for generic flat spaces ($\mathbb{R}^4$ and its toroidal compactifications) as well as some curved spaces. We briefly discuss the spectra of D-branes for both α and β families.

Keywords: Sigma Models, Superstrings and Heterotic Strings, Differential and Algebraic Geometry.

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1. Introduction and summary of results

The theories that have come to be known as $N = 2$ strings [1] date back to the period of transition of string theory from a candidate theory of hadrons to that of grand unification [2]. The worldsheet theory of $N = 2$ string has gauged $N = 2$ superconformal symmetry, which explains its name.[1] Compared with $N = 1$ superconformal field theory, which gives rise to superstring theory, it adds no new kind of fields, but another supersymmetry and an U(1) R-symmetry, for the left and right movers independently. The resulting $N = 2$ superconformal symmetry is all gauged. This theory can be thought as gauge fixed version of $N = 2$ gauged supergravity in two dimensions.

[1] To be precise, the superconformal symmetry is really $(2,2)$. However, since throughout this paper we shall only consider theories with the same amount of supersymmetries for the left and right movers, we shall simply write $N = 2$ without ambiguity. Similar conventions apply to $N = 1$ and $N = 4$.

Progress in this theory has seen fits and starts ever since. Deep connection has been found between the closed $N = 2$ strings[2] and the self-dual solution of the Einstein equations [3], between the open $N = 2$ strings and self-dual Yang-Mills [4], and between heterotic $N = 2$ strings and reduction/deformation of self-dual Yang-Mills [5]. A relation between the $N = 2$ strings and integrable models was also suggested in [3]. These have been the cause for its continuing fascination and alone justify a thorough study.

Unfortunately in reality studies of $N = 2$ strings have been somewhat disconnected from $N = 1$ superstrings, not the least because the former propagates in space with an even number of temporal dimensions. Yet there is a compelling feature of $N = 2$ strings that should appeal even to those who exclusively study superstrings: its spacetime is relative conventional, including flat $\mathbb{R}^4$ as the simplest case.[3] Unlike strings with less worldsheet spacetime supersymmetry, i.e. superstrings and bosonic strings, $N = 2$ strings typically have only finitely many degrees of freedom. In fact in the simplest setting, $\mathbb{R}^4$ of signature $(2, 2)$, it has long been established that only one scalar field propagates [1]. This makes it the simplest string theory with some physical, if not directly phenomenological, meaning. As is well-known, the infinite number of fields propagating in superstring (or bosonic string) theories render it extremely difficult to carry out analysis standard in quantum field theory unless one first takes a drastic limit that only keeps a finite numbers of fields. This often confines brilliant ideas to a state of conjectures rudimentarily tested in special limits. The simplicity of $N = 2$ strings suggests itself as a promising theoretical guinea pig where careful analysis and definite results can be obtained in a robust manner. That could be of great help to the study of superstrings or indeed any theories of strings. $N = 2$ strings seem to have no spacetime supersymmetry.[4] Issues such as the moduli fixing problem and the resolution of space-time singularities may be addressed here in a simplified set-up.

A concrete step in this program would be to understand better what all the possible $N = 2$ strings are. More than one kind of $N = 2$ string theories have been known to exist, but when we looked into this it came as a surprise that the possibilities of different $N = 2$ strings had not been systematically studied, characterized, or understood. Important insights into the geometry of the $N = 2$ string was obtained by Ooguri and Vafa in [3] where they showed that the physical scalar field of the $N = 2$ string theory describes deformation to the Kähler potential of a target space that is Kähler and the equation of motion at the tree level requires the metric be hyper-Kähler.[5] However, this cannot be the only possible $N = 2$ theory, even in $\mathbb{R}^4$. To see this, let us perform for the $N = 2$ string a standard procedure: compactify the $N = 2$ string on a circle of radius R. Like any other string theory there is a T-dual interpretation in which the spacetime involves a circle of

[2] In this work we exclusively consider critical $N = 2$ string, so we will drop the qualifier "critical." The matter part of the worldsheet conformal field theory has central charge 6. The target space dimension of the corresponding σ-model is 4.

[3] The $N = 4$ string has been argued to be equivalent to the $N = 2$ string in [6].

[4] See, however, [7].

[5] It is in this sense that the $N = 2$ string has been called a theory of self-dual gravity. This name is historical but somewhat unfortunate because among other things, only deformations of the Kähler potential with respect to a choosen complex structure is a fluctuating quantum fields in the theory, even though many other deformations of the metric can also maintain the self-duality of the Riemann tensor.

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radius α'/R. If we take R to infinity, we recover in the original description the $N=2$ string propagating in $\mathbb{R}^4$ as discussed in [3]. We shall call this theory β-string. If instead we take R to zero, the T-dual space becomes $\mathbb{R}^4$ and we again have an $N=2$ string theory propagating in $\mathbb{R}^4$, which we shall call α-string. However, α and β are not the same: they have different 3-point amplitudes [8].

In this paper we undertake a systematic classification of $N=2$ strings propagating in large class of target spaces. Our point of departure is fundamental: any $N=2$ string worldsheet theory, in the covariant formalism of the conformal field theory (CFT) approach, is obtained by gauging an $N=2$ superconformal algebra (SCA) respectively for the left and right moving chiral parts of the CFT. Therefore the physical theory is controlled both by the choice of $(2,2)$ superconformal field, with central charge $c=6$ to cancel the superconformal anomaly of the BRST ghosts, and by the choice of a $(2,2)$ SCA within that CFT to gauge. The second choice is far from unique, because many known examples of $c=6$ $N=2$ superconformal field theories (SCFT) have in fact $N=4$ SCA which allows a continuum of embeddings of $N=2$ SCA. However, different embeddings are actually physically equivalent if and only if they are related by some symmetry of the CFT. Therefore one must take care to quotient the choices of $N=2$ embeddings by the action of the field theory symmetries. To make a connection with spacetime, one considers a σ-model[6] realization of the $N=2$ SCFT. Different SCFTs correspond to different spacetimes whose geometry satisfies the β-functional equations as well as conditions for worldsheet supersymmetries. The choice of the left and right chiral $N=2$ SCAs corresponds to choosing two complex structures. Under certain simplifying assumptions, we find that the $N=2$ string theories can be broadly divided into two types, which can be interpreted in a well defined way as the generalization of the α and β theories in $\mathbb{R}^4$. This follows a careful analysis of the space of complex structures in $\mathbb{R}^4$ and the distinction is topological. It applies to curved spaces as well as flat spaces. In general, α and β strings cannot propagate in the same spaces, but flat spaces are notable and fortunate exceptions. Depending on the details of the target space geometry, there are in addition a number of continuous parameters with each type. Some of the parameters are well-known — they control the $N=2$ moduli of the SCFT and correspond to such geometric quantities as the sizes and shapes of the target space. We are concerned in this work with the rest, which specify the selection of an embedding of the left and right $N=2$ SCAs in a given CFT. They are the *parameters of gauging*. We study carefully all generic flat spaces of both euclidean and $(2,2)$ signatures and consider some prominent examples of curved spaces.

The simplifying assumptions just mentioned were made to enable a rigorous and exhaustive analysis. They are clearly and precisely stated in eq. (4.1) and eq. (5.1). Relying on them means that we cannot prove or claim that there are no other possibilities of the $N=2$ strings. Nonetheless, we wish to stress that these assumptions are based on very reasonable geometric expressions. Other possibilities, if any, would be more intricate. If they do exist, they are not likely to blur the above mentioned board division into 2 types, which is based on topological considerations. In the past literature, there had been scat-

[6]In this work we only consider *nonlinear* σ-models so we will refer to them simply as σ-models.

tered examples $N = 2$ string theories beyond the one studied in [3]. They turned out to be particular cases in the classification scheme we derive in this paper. We comment on a few of them here. [9] suggested that the SU(2) outer automorphism of $N = 4$ SCA can be used to obtain different $N = 2$ strings. This would roughly correspond to β-strings propagating on a generic $K3$, which is analyzed in section 5.2 of in the present paper. [10] and [11] considered employing elaborate variations of superspace formalism to study $N = 2$ strings. Their theories are examples of α and β types. Our unifying approach brings these examples and beyond together and organize them based on a scheme derived from the basic principles of $N = 2$ strings. Formulated solely in terms of target space geometry, it does not assume or rely on any particular $N = 2$ superspace representations. Of course, the latter can be advantageous for calculations in specific cases.

The original impetus for us to approach $N = 2$ strings was to study D-branes in the theory, in light of late developments in the $N = 1$ superstrings. It soon proved necessary to understand the closed string theory better and thus we steered toward the current study. Nonetheless D-branes are relevant to the classification of closed string theories. We have found that just as in the type-IIA and type-IIB superstrings, the $N = 2$ α and β strings admit D-branes of different dimensionalities. In this paper we mention briefly these D-brane results, leaving a full analysis to [12].

1.1 Guide to the paper

In section 2 we discuss in details the relevant aspects of flat four dimensional spaces with signatures $(4, 0)$ and $(2, 2)$. These will be used in the following sections of the paper. For those who are too impatient to read the whole section, the most important results are summarized at its beginning. In section 3 we review the facts of two-dimensional $N = 2$ and $N = 4$ compact and noncompact superconformal algebras (SCA) and their spacetime meaning. We also illustrate some embeddings of $N = 2$ SCA in $N = 4$ SCA. They are needed later for the study of $N = 2$ strings in curved spaces. People who are already familiar with the compact $N = 4$ SCAs can skip this section and return later for a quick glance at the noncompact $N = 4$ SCA in section 3.2.2 as well as section 3.3, where we present a simple extension of the $N = 4$ SCAs that is tailored to the free field realization by the σ-model on $\mathbb{R}^4$. In section 4 we will study the families of $N = 2$ strings on $\mathbb{R}^4$ and $\mathbb{R}^{2,2}$ as well as their toroidal compactifications. We propose an ansatz of simple realization of $N = 2$ SCA and find all solutions. They fall into two broad types, corresponding to the α and β strings mentioned above. We also study the continuous parameters. Then we generalize the notion of the α-β divide to general spacetimes with a precise topological definition. Section 5 concentrates on the study of curved spacetime backgrounds. A general methodology is given based on an ansatz of simple embedding. We discuss the specific case of $K3$ and outline the steps needed to study other cases, such as hyper-Kähler spaces with isometries. In section 6 we study the behavior of the $N = 2$ strings under T-duality and shows in particular that α and β strings can be related in this way. In section 7 we briefly discuss the D-branes of the $N = 2$ strings. In the appendix we summarize the notations and conventions adopted in this paper.

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Family	$\mathfrak{X}$	Rotational isometry	Parameter space of gauging
α	$\mathbb{R}^4$	$\mathrm{SU}(2) \times \mathrm{SU}(2)'$	1 point
	$\mathbb{R}^3 \times S^1$	$\mathrm{O}(3) \times \mathfrak{J}$	$(RP^2 \times RP^2)/\,\mathrm{SU}(2)$
	$\mathbb{R}^2 \times T^2$	$\mathrm{O}(2) \times \mathfrak{J}$	$(RP^2 \times RP^2)/\,\mathrm{U}(1)$
	$\mathbb{R} \times T^3$	$\mathbb{Z}_2 \times \mathfrak{J}$	$RP^2 \times RP^2$
	T^4	$\mathfrak{J}$	$RP^2 \times RP^2 \cup RP^2 \times RP^2$
β	$\mathbb{R}^4$	$\mathrm{SU}(2) \times \mathrm{SU}(2)'$	$(RP^2 \times RP^2)/\,\mathrm{SU}(2)$
	$\mathbb{R}^3 \times S^1$	$\mathrm{O}(3) \times \mathfrak{J}$	$(RP^2 \times RP^2)/\,\mathrm{SU}(2)$
	$\mathbb{R}^2 \times T^2$	$\mathrm{O}(2) \times \mathfrak{J}$	$(RP^2 \times RP^2)/\,\mathrm{U}(1)$
	$\mathbb{R} \times T^3$	$\mathbb{Z}_2 \times \mathfrak{J}$	$RP^2 \times RP^2$
	T^4	$\mathfrak{J}$	$RP^2 \times RP^2 \cup RP^2 \times RP^2$
	K3	$\emptyset$	$RP^2 \times RP^2 \cup RP^2 \times RP^2$

Table 1: Families of $N = 2$ Strings in flat euclidean and generic K3 backgrounds.

Family	$\mathfrak{X}$	Rotational isometry	Parameter space of gauging
α	$\mathbb{R}^{(2,2)}$	$\mathrm{SL}(2) \times \mathrm{SL}(2)'$	1 point
	$\mathbb{R}^{(2,1)} \times S^1$	$\mathrm{O}(2,1) \times \mathfrak{J}$	$(S^2_{1+} \times S^2_{1+})/\,\mathrm{SL}(2)$
	$\mathbb{R}^{(2,0)} \times T^{(0,2)}$	$\mathrm{O}(2) \times \mathfrak{J}$	$(S^2_{1+} \times S^2_{1+})/\,\mathrm{U}(1)$
	$\mathbb{R}^{(1,1)} \times T^{(1,1)}$	$\mathrm{O}(1,1) \times \mathfrak{J}$	$(S^2_{1+} \times S^2_{1+})/\,\mathrm{SO}(1,1)$
	$\mathbb{R} \times T^{(1,2)}$	$\mathbb{Z}_2 \times \mathfrak{J}$	$S^2_{1+} \times S^2_{1+}$
	$T^{(2,2)}$	$\mathfrak{J}$	$S^2_{1+} \times S^2_{1+} \cup S^2_{1+} \times S^2_{1+}$
β	$\mathbb{R}^{(2,2)}$	$\mathrm{SL}(2) \times \mathrm{SL}(2)'$	$(S^2_{1+} \times S^2_{1+})/\,\mathrm{SL}(2)$
	$\mathbb{R}^{(2,1)} \times S^1$	$\mathrm{O}(2,1) \times \mathfrak{J}$	$(S^2_{1+} \times S^2_{1+})/\,\mathrm{SL}(2)$
	$\mathbb{R}^{(2,0)} \times T^{(0,2)}$	$\mathrm{O}(2) \times \mathfrak{J}$	$(S^2_{1+} \times S^2_{1+})/\,\mathrm{U}(1)$
	$\mathbb{R}^{(1,1)} \times T^{(1,1)}$	$\mathrm{O}(1,1) \times \mathfrak{J}$	$(S^2_{1+} \times S^2_{1+})/\,\mathrm{SO}(1,1)$
	$\mathbb{R} \times T^{(1,2)}$	$\mathbb{Z}_2 \times \mathfrak{J}$	$S^2_{1+} \times S^2_{1+}$
	$T^{(2,2)}$	$\mathfrak{J}$	$S^2_{1+} \times S^2_{1+} \cup S^2_{1+} \times S^2_{1+}$

Table 2: Families of $N = 2$ Strings in flat (2,2) backgrounds.

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1.2 Summary of results

The calculational results are summarized in tables 1 and 2.

2. Complex geometries and symmetries of $\mathbb{R}^4$

In this work we consider exclusively critical $N = 2$ strings, therefore the target spacetime is four dimensional. In this section we derive some interesting properties of $\mathbb{R}^4$ which we will use throughout the paper. They apply globally to any flat target spaces and locally to the tangent bundle of each point on a curved target spaces. We consider the case of the metric signature $(4, 0)$ and $(2, 2)$ separately in great details.

The most important results on this section is summarized here for the impatient readers:

Given an positive definite metric in $\mathbb{R}^4$, the space of complex structures with respect to which the metric is hermitean consists of two disjoint 2-spheres. Given a metric of $(2,2)$ signature, that space consists of two disjoint hyperboloids of two sheets. In both cases, the two components are interchanged by a parity changing rotation.

The proof can be found in section 2.1.3 and section 2.2.3

2.1 Euclidean spaces

2.1.1 Vector and bi-spinor notation

The symmetry group leaving the metric invariant is O(4). Let V denote a 4 component real vector with entries V^I. The vector representation of O(4) is realized as the group of linear transformation leaving invariant the quadratic form

$$\langle V, V\rangle \equiv \mathcal{G}_{IJ} V^I V^J \tag{2.1}$$

the components of an element of its vector representation. Without loss of generality we shall set the metric to be equal to the identity matrix, so $\langle V, V\rangle$ is just the sum of the squares of V^I. Define now

$$\begin{aligned} \mathcal{V}^\alpha{}_\beta &\equiv (V^0\mathbb{I} + \imath V^i\sigma^i)^\alpha{}_\beta \\ &= \begin{pmatrix} V^0 + \imath V^3 & V^2 + \imath V^1 \\ -V^2 + \imath V^1 & V^0 - \imath V^3 \end{pmatrix}. \end{aligned} \tag{2.2}$$

Therefore

$$\det \mathcal{V} = \langle V, V\rangle\,. \tag{2.3}$$

The inverse relation is

$$V^0 = \frac{1}{2}\,\mathrm{Tr}(\mathcal{V}), \qquad V^i = \frac{-\imath}{2}\,\mathrm{Tr}(\mathcal{V}\sigma^i)\,. \tag{2.4}$$

Because V^I is are real,

$$(\mathcal{V}^\alpha{}_\beta)^* = (\sigma^2)^\alpha{}_{\alpha'}\mathcal{V}^{\alpha'}{}_{\beta'}(\sigma^2)^{\beta'}{}_\beta\,. \tag{2.5}$$

Now consider the linear transformation

$$\mathcal{V} \to U\mathcal{V}U'^{-1}\,. \tag{2.6}$$

We want to maintain the reality condition eq. (2.5) and keep $\det\mathcal{V}$ invariant. This can be arranged if

$$\begin{aligned} \sigma^2 U\sigma^2 &= U^*, \\ \det U &= 1, \end{aligned} \tag{2.7}$$

and the same for U'. This means none other than that U and U' are SU(2) matrices

$$\exp(\imath\Lambda_i\sigma^i). \tag{2.8}$$

Yet there is no condition relating U and U'. To distinguish the action of the two SU(2)'s we shall call the one acting on the second indices SU(2)$'$. We have here a realization of the $(2, 2)$ representation of SU(2) $\times$ SU(2)$'$ within the vector representation of O(4). This is hardly surprising as

$$\mathrm{SU}(2) \times \mathrm{SU}(2) = \mathrm{Spin}(4)\,. \tag{2.9}$$

Eq. (2.2) and eq. (2.4) then let us convert between the vector and the bi-spinor form of the same representation.

2.1.2 Parity

The overlap of Spin(4) and O(4) is SO(4). The center in Spin(4) that is trivial in SO(4) is $U = U' = -\mathbb{I}$. The component in O(4) that is missing from SO(4) is generated from the latter by any parity operation: reflection of an odd number of directions, e.g. $V^i \to -V^i$. The latter has a simple manifestation in $\mathcal{V}$. Consider

$$\mathcal{V}'^{\alpha}{}_{\beta} = \epsilon^{\alpha\alpha'}\mathcal{V}^{\beta'}{}_{\alpha'}\epsilon_{\beta'\beta}\,. \tag{2.10}$$

Since the ϵ tensor used here is SU(2) invariant, $\mathcal{V}'$ transform the same way as $\mathcal{V}$ and satisfies the same reality condition eq. (2.5): it is an alternative form of writing a vector in terms of bi-spinor. In fact,

$$\mathcal{V}' = V^0 - \imath V^i\sigma^i \tag{2.11}$$

so the inversion of V^i interchanges $\mathcal{V}$ and $\mathcal{V}'$. Alternatively, define

$$\mathcal{V}^{\alpha,\beta} = \mathcal{V}^{\alpha}{}_{\beta'}\epsilon^{\beta'\beta}\,. \tag{2.12}$$

Then the same operation is realized by swapping the two spinor indices:

$$\mathcal{V}^{\alpha,\beta} \leftrightarrow \mathcal{V}^{\beta,\alpha}\,. \tag{2.13}$$

Compose this with SU(2) $\times$ SU(2) has the effect of exchanging the two SU(2)'s. We shall often use the representationeq. (2.12) in this work. It transforms under SU(2) $\times$ SU(2) as

$$\mathcal{V}^{\alpha,\beta} \to U^{\alpha}{}_{\alpha'}U'^{\beta}{}_{\beta'}\mathcal{V}^{\alpha',\beta'} \tag{2.14}$$

and satisfies the reality condition

$$(\mathcal{V}^{\alpha,\beta})^* = -(\sigma^2)^{\alpha}{}_{\alpha'}(\sigma^2)^{\beta}{}_{\beta'}\mathcal{V}^{\alpha'\beta'}\,. \tag{2.15}$$

Since

$$\det \mathcal{V}^{\alpha}{}_{\beta} = \det \mathcal{V}^{\alpha,\beta} = \frac{1}{2}\epsilon_{\alpha\alpha'}\epsilon_{\beta\beta'}\mathcal{V}^{\alpha,\beta}\mathcal{V}^{\alpha',\beta'} \tag{2.16}$$

the metric in the form of tensor product of bi-spinors is

$$\frac{1}{2}\epsilon_{\alpha\beta} \otimes \epsilon_{\alpha'\beta'}\,. \tag{2.17}$$

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2.1.3 Complex structures

An almost complex structure $\mathcal{J}^I{}_J$ is a real matrix satisfying

$$\mathcal{J}\mathcal{J} = -\mathbb{I}\,. \tag{2.18}$$

It is said to make a metric $\mathcal{G}$ hermitean if

$$\mathcal{J}^\top \mathcal{G}\mathcal{J} = \mathcal{G} \tag{2.19}$$

which implies that $\mathcal{K} \equiv \mathcal{G}\mathcal{J}$ is antisymmetric

$$\mathcal{J}^\top \mathcal{G} = -\mathcal{G}\mathcal{J}\,. \tag{2.20}$$

Antisymmetric 4×4 matrices span a 6 dimensional space. A set of basis for them in bi-spinor form and satisfying the reality condition eq. (2.15) can be conveniently written in terms of tensor product:

$$\begin{aligned} 2\mathcal{K}^{[i]}_{\alpha,\alpha';\beta,\beta'} &= -\imath(\sigma^i_{\alpha\beta}) \otimes (\epsilon_{\alpha'\beta'})\,, \\ 2\mathcal{K}'^{[i]}_{\alpha,\alpha';\beta,\beta'} &= -\imath(\epsilon_{\alpha\beta}) \otimes (\sigma^i_{\alpha'\beta'})\,. \end{aligned} \tag{2.21}$$

From $\mathcal{K} = \mathcal{G}\mathcal{J}$ one finds the corresponding $\mathcal{J}$. They are

$$\begin{aligned} (J^{[i]})^{\alpha,\alpha'}{}_{\beta,\beta'} &= -\imath((\sigma^i)^\alpha{}_\beta) \otimes (\delta^{\alpha'}{}_{\beta'})\,, \\ (\mathcal{J}'^{[i]})^{\alpha,\alpha'}{}_{\beta,\beta'} &= -\imath(\delta^\alpha{}_\beta) \otimes ((\sigma^i)^{\alpha'}{}_{\beta'})\,. \end{aligned} \tag{2.22}$$

It is clear that they transform in the adjoint representations of $so(3)$ and $so(3)'$ respectively. A parity operation exchanges them as it exchanges the two SU(2)'s. Furthermore, they satisfy the following algebraic relations:

$$[\mathcal{J}^{[i]}\,,\ \mathcal{J}'^{[j]}] = 0\,, \tag{2.23}$$

and

$$\begin{aligned} \mathcal{J}^{[i]}\mathcal{J}^{[j]} &= -\delta^{ij} + \epsilon^{ijk}\mathcal{J}^{[k]}\,, \\ \mathcal{J}'^{[i]}\mathcal{J}'^{[j]} &= -\delta^{ij} + \epsilon^{ijk}\mathcal{J}'^{[k]}\,. \end{aligned} \tag{2.24}$$

Therefore they themselves are generators of $\mathrm{SU}(2)\times\mathrm{SU}(2)'$ in the $(\mathbf{2},\mathbf{2})$ representation. The nine anti-commutators $\{\mathcal{J}^{[i]}, \mathcal{J}'^{[j]}\}$ do not vanish.

An arbitrary 4×4 antisymmetric matrix $\mathcal{K} = \mathcal{G}\mathcal{J} = \mathcal{G}\sum_i(a_i\mathcal{J}^{[i]} + a'_i\mathcal{J}'^{[i]})$. The condition $\mathcal{J}\mathcal{J} = -\mathbb{I}$ dictates that

$$\sum_i (a_i)^2 + (a'_i)^2 = 1\,, \qquad a_i a'_j = 0 \tag{2.25}$$

so the only solutions are either

$$\mathcal{J} = \sum_i a_i \mathcal{J}^{[i]}, \qquad \sum_i (a_i)^2 = 1 \tag{2.26}$$

or

$$\mathcal{J} = \sum_i a'_i \mathcal{J}'^{[i]}\,, \qquad \sum_i (a'_i)^2 = 1\,. \tag{2.27}$$

Thus they are located on two disjoint spheres which are exchanged by parity operations.

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2.1.4 Stabilizer

In this course we shall often have to consider subgroup of the orthogonal group leaving invariant some/several subspaces of $\mathbb{R}^4$.

Stabilizer of V^0. Clearly, an O(3) subgroup of the O(4) symmetry leaves V^0 invariant. It is equally clear that a diagonal subgroup of SU(2) $\times$ SU(2),

$$UU' = \mathbb{I} \tag{2.28}$$

leaves V^0 invariant in eq. (2.6). This accounts for the SO(3). The full O(3) is obtained by considering an parity operation such as $V^i \to -V^i$. It corresponds to $\mathcal{V} \to -\mathcal{V}'$, or $\mathcal{V}_{\alpha\beta} \to -\mathcal{V}_{\beta\alpha}$

Stabilizer of V^0 and V^3. The obvious O(2) subgroup leaving V^0 and V^3 invariant is, in the bi-spinor form, generated by the U(1)

$$e^{\imath\theta\sigma^3} \tag{2.29}$$

plus a reflection on, say, V^2.

2.2 (2, 2) space

2.2.1 Vector and bi-spinor notation

The symmetry group leaving the metric invariant is O(2, 2). Let V denote a 4 component real vector with entries V^I. The vector representation of O(4) is realized as the group of linear transformation leaving invariant the quadratic form

$$\langle V, V\rangle \equiv \mathcal{G}_{IJ} V^I V^J \tag{2.30}$$

the components of an element of its vector representation. Without loss of generality we shall set the metric to be such that

$$\langle V, V\rangle = V^0V^0 + V^2V^2 - V^1V^1 - V^3V^3\,. \tag{2.31}$$

The relation with euclidean space is that we have V^1 and V^3 analytically continued to the imaginary axis. Define now

$$\begin{aligned} \mathcal{V}^\alpha{}_\beta &\equiv (V^0\mathbb{I} + V^1\sigma^1 + \imath V^2\sigma^2 + V^3\sigma^3)^\alpha{}_\beta \\ &= \begin{pmatrix} V^0 + V^3 & V^2 + V^1 \\ -V^2 + V^1 & V^0 - V^3 \end{pmatrix}. \end{aligned} \tag{2.32}$$

Because V^I are real,

$$\mathcal{V}^* = \mathcal{V}\,; \tag{2.33}$$

and

$$\det \mathcal{V} = \langle V, V\rangle\,. \tag{2.34}$$

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Now consider the linear transformation

$$\mathcal{V} \to U\mathcal{V}U'^{-1} . \tag{2.35}$$

We want to preserve the reality condition eq. (2.33) and keep $\det \mathcal{V}$ invariant. This can be arranged if

$$\begin{aligned} U^* &= U , \\ \det U &= 1 , \end{aligned} \tag{2.36}$$

and the same for U'. This demands that U and U' be $\mathrm{SL}(2,\mathbb{R})$ matrices

$$\exp(\Lambda_1\sigma^1 + \Lambda_3\sigma^3 + \imath\Lambda_2\sigma^2) \tag{2.37}$$

Yet U and U' are unrelated. To distinguish the action of the two $\mathrm{SL}(2,\mathbb{R})$'s we shall call the one acting on the second indices $\mathrm{SL}(2,\mathbb{R})'$. Hence we have found a realization of the $(2,2)$ representation of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ within the vector representation of $\mathrm{O}(2,2)$. Again, this is hardly surprising since

$$\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) = \mathrm{Spin}(2,2) . \tag{2.38}$$

Eq. (2.32) lets us convert between the vector and the bi-spinor forms of the same representation.

2.2.2 Parity

Unlike $\mathrm{SO}(4)$, $\mathrm{SO}(2,2)$ is not connected. Besides the component of $\mathrm{SO}(2,2)$ containing the identity, which we denote as $\mathrm{SO}_0(2,2)$, or the "proper" $\mathrm{SO}(2,2)$, there is also the "improper" one generated from $\mathrm{SO}_0(2,2)$ by the reflection of one "time" and one "space" direction. They correspond to U and U' both having determinant -1 and still being real. An example is

$$U = U' = \sigma^1 . \tag{2.39}$$

It reverses the sign of V^2 and V^3. The overlap of $\mathrm{Spin}(2,2)$ and $\mathrm{O}(2,2)$ is $\mathrm{SO}_0(2,2)$. The center in $\mathrm{Spin}(2)$ that is trivial in $\mathrm{SO}(2,2)$ is $U = U' = -\mathbb{I}$. We have just shown how to go from $\mathrm{SO}_0(2,2)$ to $\mathrm{SO}(2,2)$. The two other components in $\mathrm{O}(2,2)$ that is missing from $\mathrm{SO}(2,2)$ are generated from the latter by any parity operation: reflection of an odd number of directions, e.g. $V^i \to -V^i$. The latter has a simple manifestation in $\mathcal{V}$. Consider

$$\mathcal{V}'^{\alpha}{}_{\beta} = \epsilon^{\alpha\alpha'}\mathcal{V}^{\beta'}{}_{\alpha'}\epsilon_{\beta'\beta} . \tag{2.40}$$

Since the ϵ tensor used here is $\mathrm{SL}(2,\mathbb{R})$ invariant, $\mathcal{V}'$ transform the same way as $\mathcal{V}$ and satisfies the same reality conditioneq. (2.5): it is an alternative form of writing a vector in terms of bi-spinor. In fact,

$$\mathcal{V}' = V^0 - \imath V^i\sigma^i , \tag{2.41}$$

so the inversion of V^i interchanges $\mathcal{V}$ and $\mathcal{V}'$. Alternatively, define

$$\mathcal{V}^{\alpha,\beta} = \mathcal{V}^{\alpha}{}_{\beta'}\epsilon^{\beta'\beta} . \tag{2.42}$$

Then the same operation is realized as swapping the two spinor indices:

$$\mathcal{V}^{\alpha,\beta} \to \mathcal{V}^{\beta,\alpha} . \tag{2.43}$$

Compose this with $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ has the effect of exchanging the two $\mathrm{SL}(2,\mathbb{R})$'s. We shall often use the representation eq. (2.42) in this work. It transform under $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ as

$$\mathcal{V}^{\alpha,\beta} \to U^{\alpha}{}_{\alpha'} U'^{\beta}{}_{\beta'} \mathcal{V}^{\alpha',\beta'} \tag{2.44}$$

and has reality condition

$$(\mathcal{V}^{\alpha,\beta})^* = \mathcal{V}^{\alpha,\beta} . \tag{2.45}$$

Since

$$\det \mathcal{V}^{\alpha}{}_{\beta} = \det \mathcal{V}^{\alpha,\beta} = \frac{1}{2} \epsilon_{\alpha\alpha'} \epsilon_{\beta\beta'} \mathcal{V}^{\alpha,\beta} \mathcal{V}^{\alpha',\beta'} \tag{2.46}$$

the metric in the form of tensor product of bi-spinors is

$$\frac{1}{2} \epsilon_{\alpha\beta} \otimes \epsilon_{\alpha'\beta'} . \tag{2.47}$$

Note that even though what differentiates $(2,2)$ from $(4,0)$ is the metric, in the bi-spinor notation the forms of the metrics are actually the same as in eq. (2.17). This is because

$$\langle V, V \rangle = \det \mathcal{V} \tag{2.48}$$

holds for both cases. The difference in the metric is instead reflected by the simpler reality conditioneq. (2.45). This makes sense as the two are related by analytic continuation of V^1 and V^3 to the imaginary axis. Everything simply is real as in eq. (2.33).

2.2.3 Complex structures

An almost complex structure $\mathcal{J}^I{}_J$ is a real matrix satisfying

$$\mathcal{J}\mathcal{J} = -\mathbb{I} . \tag{2.49}$$

It is said to the make the metric $\mathcal{G}$ hermitean if

$$\mathcal{J}^\top \mathcal{G} \mathcal{J} = \mathcal{G} \tag{2.50}$$

which implies that $\mathcal{K} \equiv \mathcal{G}\mathcal{J}$ is antisymmetric

$$\mathcal{J}^\top \mathcal{G} = -\mathcal{G}\mathcal{J} . \tag{2.51}$$

This reality condition eq. (2.45) implies that both $\mathcal{K}$ and $\mathcal{J}$ are real. Real anti-symmetric 4×4 matrices are spanned by the following set of basis with real coefficients:

$$\begin{aligned} 2\mathcal{K}^{[i]}_{\alpha,\alpha';\beta,\beta'} &= (\sigma^i_{\alpha\beta}) \otimes (\epsilon_{\alpha'\beta'}) , \quad i = 1,3 ; \qquad & 2\mathcal{K}^{[2]}_{\alpha,\alpha';\beta,\beta'} &= (-\imath\sigma^2_{\alpha\beta}) \otimes (\epsilon_{\alpha'\beta'}) , \\ 2\mathcal{K}'^{[i]}_{\alpha,\alpha';\beta,\beta'} &= (\epsilon_{\alpha\beta}) \otimes (\sigma^i_{\alpha'\beta'}) , \quad i = 1,3 ; \qquad & 2\mathcal{K}'^{[2]} &= (\epsilon_{\alpha\beta}) \otimes (-\imath\sigma^2_{\alpha'\beta'}) \end{aligned} \tag{2.52}$$

Note that betweeneq. (2.21) and eq. (2.52), K^1 and K^3 has each picked up a factor of $\imath$ to become real. From $\mathcal{K} = \mathcal{G}\mathcal{J}$ one finds the corresponding $\mathcal{J}$. They are now

$$\begin{aligned}
(\mathcal{J}^{[i]})^{\alpha,\alpha'}{}_{\beta,\beta'} &= ((\sigma^i)^\alpha{}_\beta) \otimes (\delta^{\alpha'}{}_{\beta'}) \,, \quad i = 1,3 \,; \qquad (\mathcal{J}^{[2]})^{\alpha,\alpha'}{}_{\beta,\beta'} = ((-\imath\sigma^2)^\alpha{}_\beta) \otimes (\delta^{\alpha'}{}_{\beta'}) \,, \\
(\mathcal{J}'^{[i]})^{\alpha,\alpha'}{}_{\beta,\beta'} &= (\delta^\alpha{}_\beta) \otimes ((\sigma^i)^{\alpha'}{}_{\beta'}) \,, \quad i = 1,3 \,; \qquad (\mathcal{J}'^{[2]})^{\alpha,\alpha'}{}_{\beta,\beta'} = (\delta^\alpha{}_\beta) \otimes ((-\imath\sigma^2)^{\alpha'}{}_{\beta'}) \,.
\end{aligned} \tag{2.53}$$

It is clear that they transform in the adjoint representations of SO(1,2) and SO(1,2)$'$ respectively. O(2,2) elements with negative determinants exchanges them as it exchanges the two SL(2,$\mathbb{R}$)'s. Furthermore, they satisfy the following algebraic relations:

$$[\mathcal{J}^{[i]} \,, \mathcal{J}'^{[j]}] = 0 \,, \tag{2.54}$$

$$\begin{aligned}
\mathcal{J}^{[1]}\mathcal{J}^{[2]} = -\mathcal{J}^{[2]}\mathcal{J}^{[1]} &= \mathcal{J}^{[3]} \,, \\
\mathcal{J}^{[2]}\mathcal{J}^{[3]} = -\mathcal{J}^{[3]}\mathcal{J}^{[2]} &= \mathcal{J}^{[1]} \,, \\
\mathcal{J}^{[1]}\mathcal{J}^{[3]} = -\mathcal{J}^{[3]}\mathcal{J}^{[1]} &= \mathcal{J}^{[2]} \,, \\
\mathcal{J}^{[1]}\mathcal{J}^{[1]} = \mathcal{J}^{[3]}\mathcal{J}^{[3]} = -\mathcal{J}^{[2]}\mathcal{J}^{[2]} &= \mathbb{I} \,,
\end{aligned} \tag{2.55}$$

and the same for $\mathcal{J}'^{[i]}$. Therefore they themselves are generators of $so(1,2) \times so(1,2)$ in the $(\mathbf{2},\mathbf{2})$ representation. This makes sense in comparison with the euclidean case (section 2.1.3) because as we have just seen, $\mathcal{J}^{[1]}$ and $\mathcal{J}^{[3]}$ each picks up a factor of $\imath$. The nine anti-commutators $\{\mathcal{J}^{[i]}, \mathcal{J}'^{[j]}\}$ do not vanish.

An arbitrary 4×4 antisymmetric matrix $\mathcal{K} = \mathcal{G}\mathcal{J} = \mathcal{G}\sum_i(a_i\mathcal{J}^{[i]} + a'_i\mathcal{J}'^{[i]})$. The condition $\mathcal{J}\mathcal{J} = -\mathbb{I}$ dictates that

$$-(a_1)^2 + (a_2)^2 - (a_3)^2 - (a'_1)^2 + (a'_2)^2 - (a'_3)^2 = 1 \,, \qquad a_i a'_j = 0 \,, \tag{2.56}$$

so the only solutions are either

$$\mathcal{J} = \sum_i a_i\mathcal{J}^{[i]}, \qquad (a_2)^2 - (a_1)^2 - (a_3)^2 = 1 \tag{2.57}$$

or

$$\mathcal{J} = \sum_i a'_i\mathcal{J}'^{[i]}, \qquad (a'_2)^2 - (a'_1)^2 - (a'_3)^2 = 1 \,. \tag{2.58}$$

The solutions to eq. (2.57) lie on a hyperboloid of two sheets, also known as pseudo-2-sphere of index 1, S^2_1. We have thus found that the possible complex structures are parameterized by two hyperboloids of two sheets. O(2,2) elements with negative determinants exchange the two hyperboloids. The two sheets of such a hyperboloid are *simultaneously* mapped into each other by an improper SO(2,2) rotations. For example, $U = U' = \sigma^1$ does that. Note also that if a particular solution $\mathcal{J}$ lives on one sheet, then $-\mathcal{J}$ lives on the other.

The discussion here is sufficient for flat 4d space: $\mathbb{R}^4$ and its toroidal compactifications. For a curved space, there is the question of the integrability of the complex structure. The above classification of hermitian almost complex structure applies to each point on the manifold.

2.2.4 Stabilizer

In this course we shall often have to consider subgroup of the orthogonal group leaving invariant some/several subspaces of $\mathbb{R}^4$.

Stabilizer of V^0. Clearly, an $\mathrm{O}(1,2)$ subgroup of the $\mathrm{O}(2,2)$ symmetry leaves V^0 invariant. It is equally clear that an diagonal subgroup of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$,

$$UU' = \mathbb{I} \tag{2.59}$$

leaves V^0 invariant in eq. (2.35). This accounts for $\mathrm{SO}_0(1,2)$, the proper Lorentz group. $\mathrm{SO}(1,2)$ is generated when one includes the operation mentioned above that reverses the signs of V^1 and V^3. The full $\mathrm{O}(1,2)$ is obtained by considering an parity operation such as $V^i \to -V^i$. It corresponds to $\mathcal{V}_{\alpha\beta} \to -\mathcal{V}_{\beta\alpha}$.

Stabilizer of V^0 and V^2. The obvious $\mathrm{O}(2)$ subgroup leaving V^0 and V^2 invariant is, in the bi-spinor form, generated by an $\mathrm{U}(1)$

$$e^{\imath\theta\sigma^2} \tag{2.60}$$

and a reflection on, say, V^3.

Stabilizer of V^0 and V^3. The obvious $\mathrm{O}(1,1)$ subgroup leaving V^0 and V^3 invariant is in bi-spinor form generated by a "boost"

$$e^{\gamma\sigma^3} \tag{2.61}$$

plus a reflection on, say, V^2.

3. Worldsheet SCA's

$N=2$ string theory is obtained by gauging $N=2$ supersymmetry in two dimensions, i.e. a two-dimensional $N=2$ supergravity [14]. Two-dimensional $N=2$ supergravity consists of one spin 2 graviton field, two spin 3/2 gravitino fields and one spin 1 abelian gauge field. When one goes to the conformal gauge and uses the super-diffeomorphic as well as super-Maxwell symmetries to remove all the degrees of freedom of the gauge field, graviton and gravitinos, one is left with $N=2$ superconformal symmetry on the worldsheet as the residual gauge symmetry. Such worldsheet superconformal theories are distinguished by the superconformal algebra (SCA) of decoupled left and right chiral currents. For $N=2$ string, the minimal algebra is the $N=2$ SCA. It is generated by a stress-energy tensor T, two supercurrents $G^\pm$ and a $\mathrm{U}(1)$ current J.

We will consider critical $N=2$ string theories. Fixing the reparametrization invariance requires a (b,c) fermionic ghost system with $(2,-1)$ spins, two pairs $(\beta^\pm,\gamma^\pm)$ of bosonic ghost system with $(3/2,-1/2)$ spins, while fixing the $\mathrm{U}(1)$ gauge symmetry requires a $(\tilde{b},\tilde{c})$ fermionic ghost system with $(1,0)$ spins. The total ghost central charge is $c_{gh}=-6$. Consider now the matter sector. The simplest matter system arises when the $N=2$ string propagates in a flat background. It is given by an $N=2$ supersymmetric σ-model with four bosons X^I and four fermions Ψ^I. Thus, the target space is four-dimensional. The two-dimensional $N=2$ supersymmetry implies that the target space has a complex structure and therefore its signature is either $(4,0)$ or $(2,2)$. Only in the $(2,2)$ signature case the first quantized string has propagating on-shell degrees of freedom and a non-trivial dynamics in $\mathbb{R}^4$. The euclidean signature is nonetheless important as a simple reference for intuition and analysis, for curved euclidean spaces where there are highly nontrivial geometric degree of freedom, and for its possible off-shell dynamics. We will consider the algebraic structures of both signatures.

In the section we will review $N=2,4$ SCA in two dimensions and consider a bigger structure with 16 supercharges. We shall also discuss the automorphism of these SCA's, as they will be needed in this paper.

3.1 $N=2$ SCA

As noted above, the $N=2$ supersymmetry algebra is generated by a stress-energy tensor T, two supercurrents $G^\pm$ and a U(1) current J. The singular parts of their OPEs characterize their symmetry algebra:

$$
\begin{aligned}
G^+(z)G^+(w) &\sim G^-(z)G^-(w) \sim 0\,, \\
G^+(z)G^-(w) &\sim \frac{c/3}{(z-w)^3} + \frac{J(w)}{(z-w)^2} + \frac{\partial J(w)/2}{z-w} + \frac{T(w)}{z-w}\,, \\
J(z)G^\pm(w) &\sim \pm\frac{G^\pm(w)}{z-w}\,, \\
J(z)J(w) &\sim \frac{c/3}{(z-w)^2}\,, \\
T(z)G^\pm(w) &\sim \frac{\frac{3}{2}G^\pm(w)}{(z-w)^2} + \frac{\partial G^\pm(w)}{z-w}\,, \\
T(z)J(w) &\sim \frac{J(w)}{(z-w)^2} + \frac{\partial J(w)}{z-w}\,, \\
T(z)T(w) &\sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}\,.
\end{aligned}
\tag{3.1}
$$

c is the central charge normalized for the $N=0$ Virasoro algebra, i.e. for the σ-model on $\mathbb{R}^4$, $c=6$.

In term of mode operators, the above OPEs give the following $\mathbb{Z}_2$ graded Lie algebra:

$$
\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n}\,; \\
[L_m, J_n] &= -nJ_{m+n}\,; \\
[J_m, J_n] &= \frac{c}{3}m\delta_{m+n}\,; \\
[L_m, G_n^\pm] &= \left(\frac{1}{2}m - n\right)G^\pm_{m+n}\,; \\
[J_m, G_n^\pm] &= \pm G^\pm_{m+n}\,; \\
\{G_m^+, G_n^-\} &= L_{m+n} + \frac{m-n}{2}J_{m+n} + \frac{c}{6}(m^2-1)\delta_{m+n}\,; \\
\{G_m^+, G_n^+\} &= \{G_m^-, G_n^-\} = 0\,.
\end{aligned}
\tag{3.2}
$$

As given above, the abelian R symmetry of the $N = 2$ SCA generate by J_0 can be either a compact U(1) or noncompact translation or "boost." Even though the two share the same complexification, they are very different symmetries. This is reflected in the reality/Hermiticity condition imposed on the currents. Different conditions leads to different Hilbert spaces and hence different theories. In this work, as in most work on $N = 2$ strings, we consider the case of a compact U(1) symmetry, so that

$$(G^+_m)^\dagger = G^-_{-m}\,, \qquad (J_m)^\dagger = J_{-m}\,. \tag{3.3}$$

Noncompact abelian R symmetry can be studied in exactly the same vein as the analysis done in this paper.

Besides the U(1) inner automorphism generated by J, this algebra has a $\mathbb{Z}_2$ automorphism flipping the sign of J and interchanging G^+ with G^-.

$$T \to T\,, \qquad J \to -J\,, \qquad G^\pm \to G^\mp\,. \tag{3.4}$$

We call it the "conjugation" automorphism. In addition one can flip the sign of all the odd elements of the algebra all at once. The latter is present for any $\mathbb{Z}_2$ graded Lie algebra. There are other outer automorphisms. One family of outer automorphisms will be used in constructing D-brane boundary conditions in [12].

The $N = 2$ SCA has a free field realization using d pairs of complex bosons X^i and fermions Ψ^i, $i = 1, \ldots, d$. The central charge $c = 3d$. The only nonvanishing OPE's are

$$\begin{aligned} \partial X^i(z)\partial X^{\bar{j}}(w) &\sim \frac{-\mathcal{G}^{i\bar{j}}}{(z-w)^2}; \\ \Psi^i(z)\Psi^{\bar{j}}(w) &\sim \frac{\mathcal{G}^{i\bar{j}}}{z-w}\,. \end{aligned} \tag{3.5}$$

The $N = 2$ generators read

$$\begin{aligned} T &= -\mathcal{G}_{i\bar{j}} : \partial X^i \partial X^{\bar{j}} : + \frac{\mathcal{G}_{i\bar{j}}}{2}(: \partial\Psi^i\Psi^{\bar{j}} + \partial\Psi^{\bar{j}}\Psi^i :)\,, \\ J &= \mathcal{G}_{i\bar{j}} : \Psi^i\Psi^{\bar{j}} :\,, \\ G^+ &= \imath\mathcal{G}_{i\bar{j}}\Psi^i\partial X^{\bar{j}}\,, \\ G^- &= \imath\mathcal{G}_{i\bar{j}}\Psi^{\bar{j}}\partial X^i\,. \end{aligned} \tag{3.6}$$

Even though this works exactly as it is only when $\mathcal{G}_{i\bar{j}}$ is constant, $N = 2$ SCA can be realized in a $N = 2$ σ-model on any (pseudo-)Kähler manifold whose metric is Ricci-flat (plus loop corrections). Eq. (3.6) provides a useful semiclassical approximation. Since the smallness expansion parameter for such approximation is

$$\frac{\mathcal{G}_{IJ}}{\alpha'}\,, \tag{3.7}$$

one can roughly say that it is valid where the radius of spacetime is small compared to the string length. This is the regime where stringiness is small and classical differential geometry of point-set topology prevails.

When the theory is free (except orbifolds), i.e. when there is no curvature in the metric or the anti-symmetric tensor B field, the conjugation automorphism can be implemented by taking $\mathcal{G}_{i\bar{\jmath}}$ to a diagonal form and

$$\partial X^i \leftrightarrow \partial X^{\bar{i}}\,, \qquad \Psi^i \leftrightarrow \Psi^{\bar{j}}\,. \tag{3.8}$$

This is a symmetry of the theory in target spaces with a discrete isometry that acts as complex conjugation on complex coordinates, provided one does the same for the left and right movers. This last restriction is due to X. Ψ and $\tilde{\Psi}$ are independent, but X cannot be split into completely decoupled chiral and anti-chiral parts unless it is compactified: the left and right momenta are one and the same otherwise. Nonetheless, it is an automorphism of the algebra.

The relation and distinction between the symmetry of the theory and the automorphism of the chiral algebra plays an important role in discussing the family of $N = 2$ string theory in section 3. In this section we shall remark on such distinction when appropriate. As an example, note that when the target spacetime has nontrivial holonomy, eq. (3.8) ceases to be a symmetry of the algebra, because the holomorphic and anti-holomorphic tangent bundle generically are different. However, the $N = 2$ SCA itself still has a conjugation automorphism, and this leads to the mirror symmetry of Calabi-Yau manifold.

We note here that there is nothing specific in the $N = 2$ algebra that requires the metric of the target space to be euclidean in an $N = 2$ σ-model realization. $\mathcal{G}_{i\bar{\jmath}}$ can define a non-degenerate and non-definite hermitian metric. The only implication of the $N = 2$ structure on the signature of the metric is that the number of temporal and spatial dimensions must both be even.

3.2 $N = 4$

When an $N = 2$ SCFT is realized as a σ-model on target space of complex dimension 2, the SCA is automatically extended to an $N = 4$ SCA.[7] This of course corresponds to a vacuum solution of the $N = 2$ string, so we have to consider it. A typical example of the target space would be a hyper-Kähler four-manifold. The $N = 4$ SCA in fact comes in two flavors. Realized in σ-model, one corresponds to target space with $(4, 0)$ signature, such as $K3$. The other corresponds to $(2, 2)$ signature. We now consider them in turn.

3.2.1 Compact $N = 4$

The $N = 4$ algebra [15] can be obtained by supplementing an $N = 2$ algebra with two additional affine currents of charges ± 2 with respect to J. We denote them by J^+ and J^-. The OPE's among themselves are

$$\begin{aligned} J(z)\ J^{\pm}(w) &\sim \frac{\pm 2J^{\pm}}{z-w}\,, \\ J^{+}(z)\ J^{-}(w) &\sim \frac{d/2}{(z-w)^2} + \frac{J(w)}{z-w}\,, \end{aligned}$$

[7] The $N = 4$ SCA relevant to this work is what is called “small” $N = 4$.

$$J^+(z)\ J^+(z) \sim 0$$
$$J^-(z)\ J^-(w) \sim 0\,, \tag{3.9}$$

They obey the reality condition

$$(J^+_m)^\dagger = J^-_{-m}, \qquad J^\dagger_m = J_{-m}. \tag{3.10}$$

Together with J they form the affine SU(2) algebra at affine level $d/4$. Defining three hermitean chiral currents J^i

$$J^3 = \frac{1}{2}J; \qquad J^1 \pm \imath J^2 = J^\pm\,, \tag{3.11}$$

the three currents will have OPE

$$J^i(z)\ J^j(w) \sim \frac{\frac{d}{4}\delta^{ij}}{(z-w)^2} + \frac{\imath\epsilon^{ijk}J^k}{z-w}\,. \tag{3.12}$$

The supercharges must form a representation of this SU(2). Having the $N=2$ SCA embedded consistently as a sub-chiral algebra means that there are four supercharges transforming as a complex doublet of this SU(2). One can write them as $G^{\alpha,\beta}$, where α and β both range over 1 to 2. There is a reality condition

$$(G^{\alpha,\beta})^\dagger = -(\sigma^2)^\alpha{}_{\alpha'}(\sigma^2)^\beta{}_{\beta'}G^{\alpha',\beta'}\,. \tag{3.13}$$

It can be derived from the free field representation discussed below using the bi-spinor notation discussed in section 2.

$G^{\alpha,\beta}$ have the following OPE with the SU(2) currents:

$$J^i(z)G^{\alpha,\beta}(w) \sim \frac{\frac{1}{2}(\sigma^i)^\beta_\gamma G^{\alpha,\gamma}}{z-w}\,. \tag{3.14}$$

The OPE among the $G's$ themselves is

$$G^{\alpha,\beta}(z)\ G^{\gamma,\lambda}(w) \sim \frac{d\epsilon^{\alpha\gamma}\epsilon^{\beta\lambda}}{(z-w)^3} - \frac{2\epsilon^{\alpha\gamma}(\sigma^i)^{\beta,\lambda}J^i}{(z-w)^2} - \frac{\epsilon^{\alpha\gamma}(\sigma^i)^{\beta,\lambda}\partial J^i}{z-w} + \frac{\epsilon^{\alpha\gamma}\epsilon^{\beta\lambda}T}{z-w}\,. \tag{3.15}$$

The OPE's of $(T\ J)$ and of $(T\ G)$ are the conventional ones for chiral primary current of conformal weight 1 and 3/2 respectively. This completes the small $N=4$ SCA.

It is clear from the above that the compact $N=4$ SCA has two SU(2) automorphisms. One acts on the first index of $G^{\alpha,\beta}$ and we call it the outer SU(2) or $\mathrm{SU}(2)_O$. The other acts on the second index of $G^{\alpha\beta}$ and we shall call it the inner SU(2) or $\mathrm{SU}(2)_I$. $\mathrm{SU}(2)_I$ is an inner automorphism because it is generated by the zero modes of J^i. $\mathrm{SU}(2)_O$ is an outer automorphism because it is not generated by any part of the $N=4$ SCA.

We note in particular that

$$G^{1,1}(z)\ G^{2,2}(w) \sim \frac{d}{(z-w)^3} + \frac{J}{(z-w)^2} + \frac{\partial J/2}{z-w} + \frac{T}{z-w}\,. \tag{3.16}$$

So one embedding of $N=2$ SCA would have

$$J = 2J^3\,, \qquad G^+ = G^{1,1}\,, \qquad G^- = G^{2,2}\,. \tag{3.17}$$

This is of course by no means unique: any $\mathrm{SU}(2)_I \times \mathrm{SU}(2)_O$ transformation of eq. (3.16) would give an OPE of the same type, with J_3 rotated in the (3) representation of $\mathrm{SU}(2)_I$.

3.2.2 Noncompact $N = 4$

Just as there are a compact and a noncompact version of $N = 2$ SCA, the $N = 4$ SCA discussed above has a noncompact cousin which has an affine $\mathrm{SL}(2, \mathbb{R})$ as its symmetry current. As stated before, in this work we are only concerned with $N = 2$ string theories whose gauged $N = 2$ SCA is the compact one. We have just seen how it could be embedded simply in compact $N = 4$ SCA. As we shall demonstrate shortly, it could just as well be embedded in a noncompact $N = 4$ SCA. Therefore it is relevant to study the latter. In fact it appears on any σ-model with pseudo-hyper-Kähler metric, i.e. a metric with signature $(2, 2)$ that also satisfies hyper-Kähler condition.

Compared with the compact version of $N = 2$ SCA, the different OPEs are $(J^+ J^-)$, which now becomes

$$J^+(z)\, J^-(w) \sim \frac{-d/2}{(z-w)^2} - \frac{J(w)}{z-w}\,, \tag{3.18}$$

and $(G\,G)$, which becomes

$$G^{\alpha,\beta}(z)\, G^{\gamma,\lambda}(w) \sim \frac{d\epsilon^{\alpha\gamma}\epsilon^{\beta\lambda}}{(z-w)^3} - \frac{2\epsilon^{\alpha\gamma}(J^2\sigma^2 - \imath J^3\sigma^3 - \imath J^1\sigma^1)^{\beta,\lambda} J^i}{(z-w)^2} - \frac{\epsilon^{\alpha\gamma}(\sigma^i)^{\beta,\lambda}\partial J^i}{z-w} + \frac{\epsilon^{\alpha\gamma}\epsilon^{\beta\lambda} T}{z-w}\,, \tag{3.19}$$

where we have defined

$$J^2 = \frac{1}{2} J\,, \tag{3.20}$$

and J^1, J^2 by[8]

$$J^3 \pm \imath J^1 = J^\pm\,, \tag{3.21}$$

Comparing eq. (3.19) with eq. (3.9) and eq. (3.15), we see that to obtain the new OPE one merely need to add a factor of $\imath$ to $J^\pm$ each. Therefore the new OPE is consistent. However, we still require that eq. (3.10) holds. This would make no sense if the new $J^\pm$ is literally related to the old ones by a factor of $\imath$. Therefore they are really different affine Lie algebra of real coefficients that share the same complexification. Since eq. (3.9) defines affine $su(2) = so(3)$, eq. (3.18) defines affine $so(2,1) = sl(2, \mathbb{R})$ current algebra, which can be rewritten as

$$J^i(z)\, J^j(w) \sim \frac{\frac{1}{2}\eta^{ij}}{(z-w)^2} + \frac{\imath \varepsilon^{ij}_{\;k} J^k}{z-w}\,, \tag{3.22}$$

The difference from eq. (3.12) is that the signature $(2, 1)$ metric η^{ij} replaces the signature $(3, 0)$ metric δ^{ij} and that $\mathrm{SO}(2, 1)$ structure constant $\varepsilon^{ij}_{\;k}$ replaces $\mathrm{SO}(3)$ structure constant ϵ^{ijk}. The convention for η and ϵ is given in eq. (A.6) and eq. (A.7). Since $so(2, 1) = sl(2, \mathbb{R})$, this is clearly the affine $sl(2)$ current algebra.

The only other difference is

$$(G^{\alpha,\beta})^\dagger = G^{\alpha,\beta}\,. \tag{3.23}$$

[8] This is of course just a matter of convention. We make this particular choice for convenience because $\imath(\sigma^2)^\alpha{}_\beta$ is real.

That is, they are all simply hermitean, which makes sense since they are representations of $sl(2,\mathbb{R})$, which is real. All these will have a concrete realization in the free field representation given in section 3.3.

It is clear that the OPEs of the noncompact $N=4$ SCA are invariant under two $\mathrm{SL}(2,\mathbb{R})$ automorphisms. One acts on the first index of $G^{\alpha,\beta}$ and we call it $\mathrm{SL}(2,\mathbb{R})_O$. The other acts on the second index as well as on J^i and we call it $\mathrm{SL}(2,\mathbb{R})_I$. $\mathrm{SL}(2,\mathbb{R})_I$ is an inner automorphism because it is generated by the zero modes of J^i. $\mathrm{SL}(2,\mathbb{R})_O$ is an outer automorphism because it is not generated by any part of the current algebra.

We note that

$$G^+ = \frac{\epsilon_{\alpha\gamma}}{2}(\mathbb{I}-\sigma^2)^\gamma{}_\beta G^{\gamma,\beta}, \qquad G^- = \frac{\epsilon_{\alpha\gamma}}{2}(\mathbb{I}+\sigma^2)^\gamma{}_\beta G^{\gamma,\beta} \tag{3.24}$$

generate the OPE's of the compact $N=2$ SCA along with

$$J = 2J^2\,. \tag{3.25}$$

This is of course by no means unique: any $\mathrm{SL}(2,\mathbb{R})_I\times\mathrm{SL}(2,\mathbb{R})_O$ transformation of eq. (3.24) would give an OPE of the same kind, with J^2 rotated in the (3) representation of $\mathrm{SL}(2,\mathbb{R})_I$.

3.3 The fat, free $N=4$ SCA

The simplest realization of the $N=4$ SCA is the free σ-model on $\mathbb{R}^4$, which also gives the simplest $N=2$ string vacuum. It therefore deserves a careful study. Upon inspection, the most interesting feature of this model is that it contains not one, but four different realizations of $N=4$ SCA all embedded in equally simple ways. In this section we will derive this algebraic feature in detail. The geometry of $\mathbb{R}^4$ relevant to this work is studied in details in section 2. We shall make frequent uses of results derived there.

There are four real bosons and four real fermions in this theory. We can describe both of them as 2×2 matrices using the bi-spinor notation (eqs. (2.2), (2.32)). We will consider both $(4,0)$ and $(2,2)$ signature. In the bi-spinor notation, the metric has the same form for both signatures (cf. the discussion after eq. (2.48)). Therefore in both cases they have the following OPEs:

$$\begin{aligned}\partial X^{\alpha,\beta}(z)\partial X^{\gamma,\lambda}(w) &\sim \frac{-2\epsilon^{\alpha\gamma}\epsilon^{\beta\lambda}}{(z-w)^2}\,,\\ \Psi^{\alpha,\beta}(z)\Psi^{\gamma,\lambda}(w) &\sim \frac{2\epsilon^{\alpha\gamma}\epsilon^{\beta\lambda}}{z-w}\,.\end{aligned} \tag{3.26}$$

Thus the stress tensors for both cases take the form

$$T = -\frac{1}{4}\epsilon_{\alpha\gamma}\epsilon_{\beta\lambda} : \partial X^{\alpha,\beta}\partial X^{\gamma,\lambda} : +\frac{1}{4}\epsilon_{\alpha\gamma}\epsilon_{\beta\lambda} : \partial\Psi^{\alpha,\beta}\Psi^{\gamma,\lambda} : . \tag{3.27}$$

The central charge is 6.

The difference between the two types of signatures therefore lies in the reality condition on the free fields, which has important consequences on the symmetry group of the theory and automorphisms of the SCA. Indeed, eq. (3.26) and eq. (3.27) are both invariant under $\mathrm{SL}(2,\mathbb{C})$ acting on the spinor indices of $X^{\alpha,\beta}$ and $\Psi^{\gamma,\lambda}$ respectively and indepentently. That is, it has four $\mathrm{SL}(2,\mathbb{C})$ automorphisms. That symmetry is reduced differently for different types of signatures. We now consider them in turn.

3.3.1 Euclidean space

Reality. The reality condition is, from eq. (2.15)

$$
\begin{aligned}
(X^{\alpha,\beta})^\dagger &= -(\sigma^2)^\alpha{}_{\alpha'}(\sigma^2)^\beta{}_{\beta'}X^{\alpha',\beta'}\,,\\
(\Psi^{\alpha,\beta})^\dagger &= -(\sigma^2)^\alpha{}_{\alpha'}(\sigma^2)^\beta{}_{\beta'}\Psi^{\alpha',\beta'}\,,
\end{aligned}
\tag{3.28}
$$

and the same for the right chiral fermions $\tilde{\Psi}^{\theta,\gamma}$. All indices range from 1 to 2. The normalization has been chosen so that the metric for X^I and Ψ^I with O(4) vector index is identity. The vector index is related to the bi-spinor index by eq. (2.2).

Automorphisms. eq. (3.28) breaks each of the four SL$(2,\mathbb{C})$ automorphism of the free OPE down to to SU(2). We shall denote the two acting on the bosons as $\mathrm{SU}(2)_O\times\mathrm{SU}(2)_{O'}$, and the two acting on the fermions as $\mathrm{SU}(2)_I\times\mathrm{SU}(2)_{I'}$. In our convention, the unprimed groups act on the first indices and the primed groups act on the second. They all come from the rotational symmetry of euclidean $\mathbb{R}^4$: $\mathrm{SU}(2)\times\mathrm{SU}(2)'=\mathrm{Spin}(4)$. See section 2.1.1 for more detail. Since the bosons are decoupled from the fermions, they can be rotated independently. $\mathrm{SU}(2)_I$ and $\mathrm{SU}(2)_{I'}$ are both symmetries of the theory, and one can rotate the left and right chiral fermions independently. So they are further doubled. We will construct the chiral currents for them shortly. $\mathrm{SU}(2)_O$ and $\mathrm{SU}(2)_{O'}$ are symmetries of the theory but there is no chiral currents for them. One has to rotate the whole X field. However, the free chiral *algebra* eq. (3.26) and its anti-chiral counterpart do have the automorphism of rotating ∂X and $\bar\partial X$ independently, even though it is not a symmetry of the underlying *theory.*

As discussed in section 2.1.2, parity operations in O(4) exchange SU(2) and SU(2)$'$. This is reflected in the field theory as well: eq. (3.26) also automorphisms that swap the two indices of $\partial X^{\alpha,\beta}$ and/or those of $\Psi^{\gamma,\lambda}$. Again, corresponding to them are a set of symmetries for the left and right fermions Ψ's respectively and independently, but just one set for the bosons X's.

$N=4$ **SCAs.** Normal-ordered bilinear of the Ψ's give rise to 6 independent fields. They form a pair of commuting affine SU(2) currents.

$$
\begin{aligned}
J^i &= \frac{1}{8}(\sigma^i)_{\alpha\gamma}\epsilon_{\beta\lambda}:\Psi^{\alpha,\beta}\Psi^{\gamma,\lambda}:=\frac{\imath}{4}\mathcal{K}^i{}_{\alpha,\beta;\lambda,\gamma}:\Psi^{\alpha,\beta}\Psi^{\gamma,\lambda}:,\\
J'^i &= \frac{1}{8}(\sigma^i)_{\beta\lambda}\epsilon_{\alpha\gamma}:\Psi^{\alpha,\beta}\Psi^{\gamma,\lambda}:=\frac{\imath}{4}\mathcal{K}'^i{}_{\alpha,\beta;\lambda,\gamma}:\Psi^{\alpha,\beta}\Psi^{\gamma,\lambda}:.
\end{aligned}
\tag{3.29}
$$

$\mathcal{K}^i$ are defined in eq. (2.21). The OPEs of J are

$$
\begin{aligned}
J^i(z)\,J^j(w) &\sim \frac{\frac{1}{2}\delta^{ij}}{(z-w)^2}+\frac{\imath\epsilon^{ijk}J^k}{z-w}\,,\\
J'^i(z)\,J'^j(w) &\sim \frac{\frac{1}{2}\delta^{ij}}{(z-w)^2}+\frac{\imath\epsilon^{ijk}J'^k}{z-w}\,,\\
J^i(z)\,J'^j(w) &\sim 0\,.
\end{aligned}
\tag{3.30}
$$

It is easy to see that J^i and J'^i are respectively the affine currents for $\mathrm{SU}(2)_I$ and $\mathrm{SU}(2)_{I'}$.

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Since X and Ψ are free fields we can make sixteen supercharges from bilinears of them. They are conveniently grouped into four quadruplet:

$$
\begin{aligned}
G_{00}^{\alpha,\gamma} &= \frac{\imath}{2}\epsilon_{\beta\lambda}\partial X^{\alpha,\beta}\Psi^{\gamma,\lambda}\,,\\
G_{10}^{\beta,\gamma} &= \frac{\imath}{2}\epsilon_{\alpha\lambda}\partial X^{\alpha,\beta}\Psi^{\gamma,\lambda}\,,\\
G_{01}^{\alpha,\lambda} &= \frac{\imath}{2}\epsilon_{\beta\gamma}\partial X^{\alpha,\beta}\Psi^{\gamma,\lambda}\,,\\
G_{11}^{\beta,\lambda} &= \frac{\imath}{2}\epsilon_{\alpha\gamma}\partial X^{\alpha,\beta}\Psi^{\gamma,\lambda}\,.
\end{aligned}
\tag{3.31}
$$

The OPEs between the G's from different quadruplet give rise to exotic spin two fields but they are not used in the work. What is relevant is the observation that each of the four quadruplet alone is a candidate for the group of 4 supercurrent in the $N=4$ SCA. For example, the OPE's among G_{00} reproduce eq. (3.15) with J^i from eq. (3.29) as the affine SU(2) currents; so do those among G_{01}. On the other hand, the OPE's among G_{10} or G_{11} reproduce the same algebra but with J'^i in lieu of J^i. There are thus four simple ways to embed the compact $N=4$ SCA in this big algebra. The stress tensor is of course always the same: eq. (3.27). The affine currents and the supercurrents can be grouped as (J, G_{00}), (J, G_{01}), (J', G_{10}), or (J', G_{11}). It is clear that depending on the grouping, one of $\mathrm{SU}(2)_I$ and $\mathrm{SU}(2)_{I'}$ becomes the inner SU(2) of the $N=4$ SCA, while one of $\mathrm{SU}(2)_O$ and $\mathrm{SU}(2)_{O'}$ becomes the outer SU(2).

3.3.2 (2, 2)

Reality. The reality condition is, from eq. (2.45)

$$
(X^{\alpha,\beta})^\dagger = X^{\alpha,\beta}\,, \qquad (\Psi^{\alpha,\beta})^\dagger = \Psi^{\alpha,\beta}\,. \tag{3.32}
$$

Automorphisms. eq. (3.32) breaks each of the four $\mathrm{SL}(2,\mathbb{C})$ automorphism of the free OPE down to to $\mathrm{SL}(2,\mathbb{R})$. We shall denote the two acting on the bosons as $\mathrm{SL}(2,\mathbb{R})_O \times \mathrm{SL}(2,\mathbb{R})_{O'}$, and the two acting on the fermions as $\mathrm{SL}(2,\mathbb{R})_I \times \mathrm{SL}(2,\mathbb{R})_{I'}$. In our convention, the unprimed groups act on the first indices and the primed groups act on the second. They all come from the rotational symmetry of (2, 2) space: $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})' = \mathrm{Spin}(2,2)$. See section 2.2.1 for more detail. Since the bosons are decoupled from the fermions, they can be rotated independently. $\mathrm{SL}(2,\mathbb{R})_I$ and $\mathrm{SL}(2,\mathbb{R})_{I'}$ are both symmetries of the theory, and one can rotate the left and right chiral fermions independently. So they are further doubled. We will construct the chiral currents for them shortly. $\mathrm{SL}(2,\mathbb{R})_O$ and $\mathrm{SL}(2,\mathbb{R})_{O'}$ are symmetries of the theory but there is no chiral currents for them. One has to rotate the whole X field. However, the free chiral *algebra* eq. (3.26) and its anti-chiral counterpart do have the automorphism of rotating ∂X and $\bar{\partial} X$ independently, even though it is not a symmetry of the underlying *t*heory.

As discussed in section 2.2.2, parity operations in O(2, 2) exchange $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SL}(2,\mathbb{R})'$. This is reflect in the field theory as well: eq. (3.26) also admits automorphisms that swap the two indices of $\partial X^{\alpha,\beta}$ and/or those of $\Psi^{\gamma,\lambda}$. Again, corresponding to them are a set of symmetries for the left and right fermions Ψ's respectively and independently, but just one set for the bosons X's.

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$N=4$ **SCAs.** Normal-ordered bilinear of the Ψ's give rise to 6 independent fields. They form a pair of commuting affine $\mathrm{SL}(2,\mathbb{R})$ currents.

$$\begin{aligned} J^i &= \frac{\imath}{4}\mathcal{K}^i{}_{\alpha,\beta;\lambda,\gamma} : \Psi^{\alpha,\beta}\Psi^{\gamma,\lambda} : , \\ J'^i &= \frac{\imath}{4}\mathcal{K}'^i{}_{\alpha,\beta;\lambda,\gamma} : \Psi^{\alpha,\beta}\Psi^{\gamma,\lambda} : . \end{aligned} \tag{3.33}$$

$\mathcal{K}^i$ are defined in eq. (2.52). The OPEs of J are

$$\begin{aligned} J^i(z)\, J^j(w) &\sim \frac{\frac{1}{2}\eta^{ij}}{(z-w)^2} + \frac{\imath\varepsilon^{ij}_{k} J^k}{z-w} , \\ J'^i(z)\, J'^j(w) &\sim \frac{\frac{1}{2}\eta^{ij}}{(z-w)^2} + \frac{\imath\varepsilon^{ij}_{k} J'^k}{z-w} , \\ J^i(z)\, J'^j(w) &\sim 0 . \end{aligned} \tag{3.34}$$

Comparing with eq. (3.30) it is clear that J^i and J'^i are respectively the affine currents for $\mathrm{SL}(2,\mathbb{R})_I$ and $\mathrm{SL}(2,\mathbb{R})_{I'}$.

Since X and Ψ are free fields we can make sixteen supercharges from bilinear of them. They are conveniently grouped into four quadruplet:

$$\begin{aligned} G^{\alpha,\gamma}_{00} &= \frac{\imath}{2}\epsilon_{\beta\lambda}\partial X^{\alpha,\beta}\Psi^{\gamma,\lambda} , \\ G^{\beta,\gamma}_{10} &= \frac{\imath}{2}\epsilon_{\alpha\lambda}\partial X^{\alpha,\beta}\Psi^{\gamma,\lambda} , \\ G^{\alpha,\lambda}_{01} &= \frac{\imath}{2}\epsilon_{\beta\gamma}\partial X^{\alpha,\beta}\Psi^{\gamma,\lambda} , \\ G^{\beta,\lambda}_{11} &= \frac{\imath}{2}\epsilon_{\alpha\gamma}\partial X^{\alpha,\beta}\Psi^{\gamma,\lambda} . \end{aligned} \tag{3.35}$$

The OPEs between the G's from different quadruplet give rise to exotic spin two fields but they are not used in the work. What is relevant is the observation that each of the four quadruplet alone is a candidate for the group of 4 supercurrent in the $N=4$ SCA. For example, the OPE's among G_{00} reproduce eq. (3.19) with J^i from eq. (3.33) as the affine $\mathrm{SL}(2,\mathbb{R})$ currents; so do those among G_{01}. On the other hand, the OPE's among G_{10} or G_{11} reproduce the same algebra but with J'^i in lieu of J^i. There are thus four simple ways to embed the noncompact $N=4$ SCA in this fat, free noncompact $N=4$ algebra. The stress tensor is of course always the same: eq. (3.27). The affine currents and the supercurrents can be grouped as (J,G_{00}), (J,G_{01}), (J',G_{10}), or (J',G_{11}). It is clear that depending on the grouping, one of $\mathrm{SL}(2,\mathbb{R})_I$ and $\mathrm{SL}(2,\mathbb{R})_{I'}$ becomes the inner $\mathrm{SL}(2,\mathbb{R})$ of the $N=4$ SCA, while one of $\mathrm{SL}(2,\mathbb{R})_O$ and $\mathrm{SL}(2,\mathbb{R})_{O'}$ becomes the outer $\mathrm{SL}(2,\mathbb{R})$.

4. α, β, etc.

4.1 Principle and recipe

In this section we will study the families of $N=2$ strings on $\mathbb{R}^4$ and $\mathbb{R}^{2,2}$ as well as its toroidal compactifications. We propose an ansatz of a simple realization of the $N=2$ SCA and find all solutions. They fall into two broad disjoint classes, which we respectively name as α and β strings. We will study their parameters of gauging. Finally we will generalize the notion of the α-β divide to general spacetime with a precise and topological definition.

4.1.1 Pre-theory and gauge theory

$N = 2$ SCFT with a central charge of 6 is the most usual starting point for studying $N = 2$ string theory, but the two are really different theories, even from a worldsheet point of view. To string theory, the gauged chiral algebra is the remnant of the worldsheet gauge symmetry. Residual gauge symmetry represents the redundancy in the CFT description of the system not removed by the conformal gauge fixing. The true physical content of the $N = 2$ string worldsheet theory therefore resides in the quotient of the CFT by the $N = 2$ SCA.[9] If there are different ways to embed $N = 2$ SCA in its chiral algebra, to specify an $N = 2$ string theory we have to specify which one to gauge. Different embedding, however, does not necessarily leads to inequivalent quotients. If one embedding is related to another by a symmetry of the theory, their quotient will related by a relabeling of fields corresponding to that symmetry and cannot represent any physical difference. Otherwise, the resulting theories are really physically distinct.

The above consideration leads to a principle that is applicable to all string theories and in fact all theories obtained from another, which we shall call the *pre-theory*, by gauging some symmetry algebra therein:

> *Pre-theory is supposed to be a redundant description of the physical degree of freedom and dynamics. The redundancy is expressed by the gauge symmetry.*
>
> $$\textit{Physical Theory} = \frac{\textit{Pre-theory}}{\textit{Gauge Symmetry}}$$
>
> *If we are given a pre-theory and asked to find the physical theory, we have to known not only the intrinsic property of the gauge symmetry, i.e. what it is, but also its extrinsic property as a part of the pre-theory, i.e. how it is realized — how it acts. Both information together determine what is redundant and what is not in the pre-theory. One pre-theory and one type of gauge symmetry can thus lead to a family of physically distinct theories. The space of such theories is given by the quotient of the space of all realization of the gauge symmetry in the pre-theory, quotiented by all the symmetry of the theory. In other words, the space of physically distinct theories is the space of orbits by the action of all symmetries of the pre-theory on the space of all realizations of the gauge symmetry. We call it the* parameter space of gauging.

What is special about the $N = 2$ string is that it readily provides examples of nontrivial family of physically inequivalent embedding due to both its rich structure and some field-theoretic "coincidence." Although not logically necessary, the most studied pre-theories of $N = 2$ strings have space(-time) interpretation because they have more geometric as well as physical relevance. They are the $N = 2$ σ-models with some $4d$ target space $\mathfrak{X}$. The dimension 4 is due to the relation between central charge, required to be 6 by anomaly cancellation, and the dimension of the spacetime. It happens that the $N = 2$ superconformal invariance for such models often implies the existence of the bigger $N = 4$

[9] The word quotient here has a precise meaning in the language of hamiltonian formulation. It is tantamount to imposing constraints as in bosonic string and superstring theories.

SCA. There is not just one, but a continuous range of possible embeddings of the $N=2$ SCA into the $N=4$. Furthermore, the $N=2$ string propagating in flat 4d spaces has the "fat, free" $N=4$ SCA, which yields even more choices for embedding an $N=2$ SCA. In general, exhaustion of all embeddings of a particular algebra in a bigger algebra may require a massive effort when the algebras under question are infinite-dimensional. However we found that by restricting ourselves to *simple* realizations or embeddings of the $N=2$ SCA, to be defined precisely below,[10] we can completely classify the space of all physically distinct $N=2$ string theories and they are nontrivial. We may add that although we do not claim their nonexistence, we are not yet aware of any realizations that are not simple and hence fall out of our classification. In any case the simple realizations are the one with the most direct geometric interpretation.

4.1.2 Symmetries of a σ-model

The first step in the classifications we undertake is to identify all possible realizations of $N=2$ SCA satisfying our assumption of "simplicity." After that we have to find out how the symmetries of the CFT acts on them so we can find the orbits of this action. The first step is determined by the chiral algebra in which we are to embed the $N=2$ SCA and is the more straightforward step because our simplicity condition is fairly restrictive. The second step is subtler and here we give a detailed general discussion.

In a supersymmetric σ-model the bosonic fields X^I define a map from the two-dimensional worldsheet into the target space $\mathfrak{X}$; while the fermions Ψ^I defines a section in the pull back of the tangent bundle of the $\mathfrak{X}$ induced by the map X. The symmetries of a model as geometric as such are intimately related to the geometry of the target space, but the very different nature of X^I and Ψ^I brings about rather different relations. Every isometry of the target space, i.e. coordinate transformations that leaves the metric invariant, corresponds to a symmetry acting on X^I. The induced action on the tangent bundle corresponds to an accompanying action on Ψ. There are also symmetries on the Ψ alone. In a target space interpretation they would be a rotation that leave the metric invariant, i.e. an element of the orthogonal group of appropriate signature. Furthermore this rotation needs to leave the affine connection on $\mathfrak{X}$ invariant; otherwise it would not leave the σ-model action invariant. So the kind of symmetry group is the commutant of the holonomy group of $\mathfrak{X}$ within the appropriate orthogonal group. If, for example, $\mathfrak{X}$ is a generic Kähler manifold, the symmetry for the fermions would be an U(1). In fact, since the left and right chiral fermions on the worldsheet decouple, they can be rotated independently, the number of symmetries is doubled. Contrast this with X. The left and right moving parts of X do not decouple. Therefore the symmetries acting on X^I is in one to one correspondence with the isometries of $\mathfrak{X}$ which gives rise to them.[11] σ-model on group manifold with the appropriate WZNW term is such an example.

[10] These embeddings are explicitly given in eq. (4.1) for the flat space, and eq. (5.1) and eq. (5.11) for more general cases.

[11] There are cases, for example with the target space a torus, when there are additional symmetry for the X's. However, they do not have a direct geometric interpretation in the target space. For torus, a T-duality links them with isometry of the T-dual target space.

Finally, when the worldsheet theory has parity invariance, exchanging left and right mover is a symmetry. For a σ-model, standard worldsheet parity symmetry holds if and only the antisymmetric tensor B field vanishes. If so, we should mod out the exchange symmetry of $\mathcal{J}$ and $\tilde{\mathcal{J}}$ to obtain the space of gauge parameters without duplication. We note that sometimes a modified version of worldsheet parity transformation exists, even though the standard parity symmetry is broken by the presence of B field. The unbroken symmetry is a parity transformation plus a rearrangement of say the right movers. This happens on compactification over a torus with B field that is T-dual to a non-rectangular torus without B field. One should take all such symmetry, indeed all symmetries of the CFT into account when determining the parameter space of gauging for a specific case, although the preceding list seems to cover all known cases.

These considerations lead to the following recipe for finding different $N = 2$ string theories.

> *Find all realizations of the $N = 2$ SCA in the pre-theory. Quotient out by the symmetries only acting on the fermions, which is the commutant of the holonomy group of the target space. This can be done independently for the left and right movers and gives the same reduced space of embedding for each. The total reduced space is their product. Next quotient out by the isometries of the target space $\mathfrak{X}$. They often act on the left and right chiral algebra simultaneously and therefore quotienting by it may intertwines the two reduced spaces of embeddings. Finally, if the theory has some symmetry involving worldsheet parity we should mod it out as well. This will exchange the left and right movers' reduced spaces of gauging parameters up to some transformation.*

As we consider different target space geometries, the number of symmetries affecting the X and Ψ vary from case to case. This affects the range of distinct $N = 2$ string vacua in two ways. On the one hand the number of ways to embed an $N = 2$ SCA in the chiral algebra changes. Generally it decrease when the theory becomes less symmetric. On the other hand the number of symmetries that relates these different embeddings also decrease. In this section we will analyze the case of flat space thoroughly. It is not only the simplest but perhaps also the most important physically.

4.2 Simple realizations of the $N = 2$ SCA

As a first step we want to find the space of different embeddings of the $N = 2$ SCA in the free σ-model on $\mathbb{R}^4$. Because the infinite-dimensional nature of the operator algebra a *rigorous* and *exhaustive* survey of all possible embeddings of the $N = 2$ SCA in a given theory is technically rather difficult, even for a free theory. So here we shall instead prove a result for an embedding of the $N = 2$ SCA satisfying certain simplifying assumptions. Fortunately, this turns out to give nontrivial examples already. Other cases, if they exist, would be more subtle. The analysis takes advantage of the fact that the theory is free. When one is dealing with an interacting CFT, such as in curved spacetime section 5, one has to rely on the abstract knowledge of the symmetry currents. The methodology will be developed therein.

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As we reviewed in section 2, the $N = 2$ SCA has a free field realization eq. (3.6). Generalizing this we shall consider realizations of $N = 2$ SCA currents in terms of normal ordered bilinears of free fields.

$$\begin{aligned}
T &= -\frac{1}{2}\mathcal{G}_{IJ} :\partial X^I \partial X^J: + \frac{\mathcal{G}_{IJ}}{2}(:\partial\Psi^I\Psi^J + \partial\Psi^J\Psi^I:)\,, \\
J &= \frac{\imath}{2}\mathcal{K}_{IJ} :\Psi^I\Psi^J:, \\
G^+ &= \imath m_{IJ}\ \Psi^I\partial X^J\,, \\
G^- &= \imath m^*_{IJ}\ \Psi^I\partial X^J\,.
\end{aligned} \tag{4.1}$$

We call such free field realization "simple." The (conjugation) relation between $G^\pm$ is required by the compactness of the U(1) symmetry associated with J.

Let m_{IJ} be an arbitrary $D \times D$ complex matrix. $D = 2d$ is the real dimensions of the target space. The OPE of $(G^+\ G^+)$ and $(G^-\ G^-)$ should vanish. This leads to the following conditions on m:

$$m\mathcal{G}^{-1}m^\top = m^\top\mathcal{G}^{-1}m = 0\,. \tag{4.2}$$

OPE of $(G^+\ G^-)$ is given by eq. (3.1), so we find

$$m\mathcal{G}^{-1}m^\dagger + m^*\mathcal{G}^{-1}m^\top = m^\top\mathcal{G}^{-1}m^* + m^\dagger\mathcal{G}^{-1}m = \mathcal{G}\,, \tag{4.3}$$

and

$$\imath\mathcal{K} = (m\mathcal{G}^{-1}m^\dagger - m^*\mathcal{G}^{-1}m^\top)\,. \tag{4.4}$$

The OPE of $(J\ G^\pm)$ results in the conditions

$$\imath\mathcal{K}\mathcal{G}^{-1}m = m, \qquad \imath\mathcal{K}\mathcal{G}^{-1}m^* = -m^*\,. \tag{4.5}$$

and the OPE of $(J\ J)$ gives

$$\mathrm{Tr}(\mathcal{K}g^{-1})^2 = D\,. \tag{4.6}$$

Define $\mathcal{M} = \mathcal{G}^{-1}m$. The above condition on m can be rewritten as

$$\begin{aligned}
\mathcal{M}\mathcal{G}^{-1}\mathcal{M}^\top &= \mathcal{M}^\top\mathcal{G}^{-1}\mathcal{M} = 0\,, \\
\mathcal{M}\mathcal{G}^{-1}\mathcal{M}^\dagger + \mathcal{M}^*\mathcal{G}^{-1}\mathcal{M}^\top &= \mathcal{G}^{-1}\,, \\
\mathcal{M}^\top\mathcal{G}\mathcal{M}^* + \mathcal{M}^\dagger\mathcal{G}\mathcal{M} &= \mathcal{G}\,.
\end{aligned} \tag{4.7}$$

and

$$\imath\mathcal{K}\mathcal{M} = \mathcal{G}\mathcal{M}\,, \qquad \imath\mathcal{K}\mathcal{M}^* = -\mathcal{G}\mathcal{M}^*\,, \tag{4.8}$$

with

$$\imath\mathcal{G}^{-1}\mathcal{K}\mathcal{G}^{-1} = \mathcal{M}\mathcal{G}^{-1}\mathcal{M}^\dagger - \mathcal{M}^*\mathcal{G}^{-1}\mathcal{M}^\top\,. \tag{4.9}$$

One then finds that

$$(\mathcal{M} + \mathcal{M}^*)\mathcal{G}^{-1}(\mathcal{M} + \mathcal{M}^*)^\top = \mathcal{G}^{-1} \tag{4.10}$$

Therefore $\mathcal{M}+\mathcal{M}^*$ leaves the metric invariant: it must be an element of O(4,0) or O(2,2). Since

$$G^+ = \imath \mathcal{M}^I{}_J \ \Psi_I \partial X^J \,, \qquad G^- = \imath \mathcal{M}^{*I}{}_J \ \Psi_I \partial X^J$$

we can assume without loss of generality that $\mathcal{M}+\mathcal{M}^* = \mathbb{I}$ by rotating Ψ^J,[12] Doing so eliminates some of the symmetries that relate different embeddings. Therefore we can write

$$\mathcal{M} = \frac{1}{2}(\mathbb{I} + \imath \mathcal{J}) \tag{4.11}$$

for some real matrix $\mathcal{J}$, i.e.

$$\mathcal{J} = -\imath(\mathcal{M} - \mathcal{M}^*) \,. \tag{4.12}$$

One infers from eq. (4.9) that

$$\mathcal{K} = \frac{1}{2}(\mathcal{G}J - J^\top \mathcal{G}) \,. \tag{4.13}$$

It then follows that $\mathcal{J}$ satisfy

$$\mathcal{J}^\top \mathcal{G} \mathcal{J} = \mathcal{G} \,, \qquad \mathcal{J}\mathcal{J} = -\mathbb{I} \,. \tag{4.14}$$

Thus $\mathcal{J}$ is a complex structure hermitian with respect to the metric $\mathcal{G}$. Furthermore it follows either from eq. (4.7) or from eq. (4.13) and eq. (4.8) that

$$\mathcal{K} = \mathcal{G}\mathcal{J} \tag{4.15}$$

is the Kähler form. All the other equations are satisfied.

Therefore we want to find all possible complex structures $\mathcal{J}$ that makes $\mathcal{G}$ hermitian. This mathematical problem has already been solved in section 2.1.3 and section 2.2.3. For (4,0) metric the solutions lies on two disjoint S^2 while for (2,2) metric the solution lies on two disjoint S^2_1. Eq. (4.1) and eq. (4.15) then determine the affine U(1) current in terms of the choice for the complex structures. Moreover upon comparing eq. (3.29) with eq. (2.21) and eq. (3.29) witheq. (2.21) one sees that the pair of affine SU(2) (SL(2,$\mathbb{R}$)) currents is directly to the SU(2) (SL(2,$\mathbb{R}$)) algebra complex structures hermitean with respect to a (4,0) ((2,2)) metric. To make an $N = 2$ SCA the affine U(1) has to come from either of the pair, but not a linear combination of both. Note also that $\mathcal{J}$ and $-\mathcal{J}$ defines the same $N = 2$ SCA: it corresponds to the conjugation automorphism of eq. (3.4). Quotiented by this $\mathbb{Z}_2$, S^2 becomes the real projective plane RP^2, and S^2_1 becomes the upper sheet of its former self, S^2_{1+}. Therefore the space of choices is actually reduced by half to two disjoint RP^2 for (4,0) metrics, and two disjoint S^2_{1+} for (2,2) metrics. We denote them by $RP^2 \cup RP^{2\prime}$ and $S^2_{1+} \cup S^2_{1+}{}'$ respectively, where the meaning of ' is hopefully clear. Note also that this space is already reduced from the the total space of different embeddings because we have used the symmetry of the theory that rotate Ψ^I to make $\mathcal{M}+\mathcal{M}^* = \mathbb{I}$.

Since the worldsheet theory includes both the left and right movers, each side has this same amount of choices, and the total space of choices is the square of what we have just

[12] Note since the left and right chiral fermions can be rotated independently, this assumption can be made for both movers independently and simultaneously.

found. So all in all, we have found that partially reduced parameter space of gauging for euclidean flat spaces to be

$$(RP^2 \cup RP^{2\prime}) \times (RP^2 \cup RP^{2\prime}) = \left(RP^2 \times RP^{2\prime} \cup RP^{2\prime} \times RP^2\right) \cup \\ \cup \left(RP^2 \times RP^2 \cup RP^{2\prime} \times RP^{2\prime}\right). \qquad (4.16)$$

For flat $(2,2)$ spaces it is

$$(S^2_{1+} \cup S^2_{1+}{}') \times (S^2_{1+} \cup S^2_{1+}{}') = \left(S^2_{1+} \times S^2_{1+}{}' \cup S^2_{1+}{}' \times S^2_{1+}\right) \cup \left(S^2_{1+} \times S^2_{1+} \cup S^2_{1+}{}' \times S^2_{1+}{}'\right). \qquad (4.17)$$

Already we can define what we mean by α and β strings by looking at the RHS. β strings choose $\mathcal{J}$ and $\tilde{\mathcal{J}}$ from the same S^2 or S^2_{1+}, while α strings choose from different ones. Hence they respectively belong to the sets grouped in the first and second pairs of parentheses in the r.h.s. of the equations above. From this we now have to find the orbits of the symmetries acting on X^I, Certainly this does not only depend on what the symmetry group is but also how it acts on the reduced parameter space. We will start analyzing case by case noncompact and compact flat spaces. One remark is warranted here. As mentioned in the recipe given in section 4.1.2, if the worldsheet theory has some symmetry involving worldsheet parity change, the left and right chiral parts can be interchanged. We should then quotient the product above by it. The only term in this free field theory that can violate parity is the presence of a nonvanishing but flat B field. It has no effect on the OPEs and therefore has not entered into the discussion until now. However, even with a nonvanishing B field it is possible to have a symmetry that involves the flip of worldsheet parity *and* a rotation of, say, the right moving fields, while each alone is not a symmetry of the worldsheet theory. We will discuss this important point further when we discuss toroidal compactification of two or more directions in section 4.4 because only in these situations can there be background for $B \neq 0$ that is not a gauge artifact.

4.3 Choice mod symmetry: α versus β

For $\mathbb{R}^4$, the rotational isometry to quotient is $\mathrm{O}(4)$ or $\mathrm{O}(2,2)$, depending on the signature of the metric. We study them below separately.

4.3.1 Euclidean

Since we have already fixed $M + M^*$ to be $\mathbb{I}$, the symmetries that are left are the rotations that leave this invariant. As

$$\partial X^I \to \Lambda^I{}_J \partial X^J\,, \qquad \Psi^I \to \Gamma^I{}_J \Psi^J\,, \qquad \mathcal{M}^{\mathcal{I}}{}_J = (\mathbb{I} + \imath J)^I{}_J \to (\Gamma M \Lambda)^I{}_J\,, \qquad (4.18)$$

what is allowed is a diagonal action of the rotations on Ψ and X:

$$\Gamma\Lambda = \mathbb{I}\,. \qquad (4.19)$$

The need to involve rotations on X means that one cannot independently perform the allowed rotations separately for left and right movers. Included in the rotations are parity operations that swap $\mathcal{J}$ and $\mathcal{J}'$, as well as arbitrary $\mathrm{SO}(4)$ action. Therefore we may fix $\mathcal{J}$ to be, say, $\mathcal{J}^{[3]}$. This choice is still invariant under $\mathrm{SU}(2)'$ and a $\mathrm{U}(1)$ subgroup of $\mathrm{SU}(2)$ generated by J^3.

Now look at the right movers. We no longer have the freedom to pick which sphere of complex structure to use for $\tilde{\mathcal{J}}$, because doing so requires swapping the two SU(2) indices of $X^{\alpha,\beta}$ (section 2.1.2), but this would upset the choice already made for the left mover. Therefore just as argued in the section 3.1,

> *there are two broad types of $N = 2$ strings in $\mathbb{R}^4$. In one the left and right chiral* U(1) *currents respectively use complex structures on two* different *spheres, and we call it the α-string; in the other they are from the* same *sphere, and we call it the β-string.*[13]

This definition is compatible with the earlier grouping of the components of the partially reduced parameter space of gauging.

For α-string, we can rotate $\tilde{\mathcal{J}}$ to a particular point of $RP^{2'}$ by an SU(2)$'$ rotation, which does not affect the already made choice for $\mathcal{J}$. So there is no more physical inequivalent choices to be made.

For β-string, it is impossible to fix $\tilde{\mathcal{J}}$ completely because the only rotation left is the U(1) that rotates $\tilde{\mathcal{J}}$ around $\mathcal{J}$. It leaves $\mathcal{J}$ invariant. Since $\mathcal{J}$ and $\tilde{\mathcal{J}}$ use the same sphere of complex structures it makes sense to compare them, i.e. measure their difference by the angle between them. In fact this parameterizes the orbits of $\tilde{\mathcal{J}}$ under the residual U(1). Therefore the choices is represented by a variable θ ranging from 0 to π, measuring the angle between the left and right moving $\mathcal{J}$ on the sphere. The range of θ is further reduced to $0\ldots\pi 2$ because, as mentioned above, inequivalent choices lie on $RP^2 = S^2/Z_2$ instead of S^2. We call the $N = 2$ strings theories with this choice as type β_θ.

Starting with the partially reduced parameter space in eq. (4.16), what we have just shown is in algebraic terms

$$\frac{(RP^2 \cup RP^{2\prime}) \times (RP^2 \cup RP^{2\prime})}{\mathbb{Z}_2 \times \mathrm{O}(4)} = \{1 \text{ point}\} \cup \frac{RP^2 \times RP^2}{\mathrm{SU}(2)} . \tag{4.20}$$

On the r.h.s. first part of the union corresponds to the α string; the second to the β string. The $\mathbb{Z}_2$ on the l.h.s. is the worldsheet parity which exchanges $\mathcal{J}$ and $\tilde{\mathcal{J}}$. There is no relevant B field background in empty R^4 and so it is a symmetry. Without it the r.h.s. would have been doubled.

4.3.2 (2, 2)

Much of the analysis for (4, 0) signature carries over to (2, 2) signature, with SL(2, $\mathbb{R}$) replacing SU(2), so we shall be brief and emphasize only the differences. By diagonal O(4, 4) action on X and Ψ, we can again fix

$$\mathcal{J} = \mathcal{J}^{[2]} . \tag{4.21}$$

For the right movers, there is again a discrete choice between putting $\tilde{\mathcal{J}}$ on the same hyperboloids or the other one. β-string chooses the same and α-string chooses different.

[13] It should be obvious that what matters is how one complex structure differ relative to the other. A parity symmetry operation on X cause both sides to hop to the other sphere.

For α-string, the residual $\mathrm{SL}(2,\mathbb{R})'$ can fix $\tilde{\mathcal{J}}$ once one decides on which sheet of the second hyperboloid it should lie. However, the choice between the two sheets of the same hyperboloid is not really a choice for different embeddings of $N = 2$ SCA. It corresponds to the $\mathbb{Z}_2$ conjugation automorphism of $N = 2$ SCA eq. (3.4) and hence gives the same $N = 2$ SCA. Therefore the physically distinct choice is unique.

For β-string, the same argument says that we can put $\mathcal{J}$ and $\tilde{\mathcal{J}}$ on the same sheet of their hyperboloid. By using $\mathrm{SL}(2,\mathbb{R})$ we can fix $\mathcal{J}$ to, say the point $r = 0$ on S^2_1+. See section B for an explanation of the notation. The subgroup of $\mathrm{SL}(2,\mathbb{R})$ that leaves $\mathcal{J}$ invariant is the compact U(1) generated by its zero mode. Its orbits are parameterized by r, which measures the separation between $\mathcal{J}$ and $\tilde{\mathcal{J}}$. It ranges from 0 to ∞. We call $N = 2$ string theory parameterized by it as type β_r.

Starting with the partially reduced parameter space in eq. (4.17), what we have just shown is in algebraic terms

$$\frac{(S^2_{1+} \cup S^2_{1+}{}') \times (S^2_{1+} \cup S^2_{1+}{}')}{\mathbb{Z}_2 \times \mathrm{O}(4)} = \{1 \text{ point}\} \cup \frac{S^2_{1+} \times S^2_{1+}}{\mathrm{SL}(2,\mathbb{R})} \,. \tag{4.22}$$

Again the first part of the union corresponds to the α string; the second to the β string. The meaning of the $\mathbb{Z}_2$ on the l.h.s. and its effect on the r.h.s. exchanging are the same as in the euclidean case.

4.4 Toroidal compactification

Now we go one step up in complexity and consider toroidal compactifications. The target space remains flat, so the theory remains free. The chiral algebra therefore does not change, and it still contains the fat, free $N = 4$ SCA. Therefore the analysis of section 4.2 remains applicable, and the number of ways of embedding an $N = 2$ SCA remains the same. However while the symmetries acting just on the Ψ's do not change because they are not sensitive to the compactification, rotation symmetries on the X's is broken by it. Therefore the number of distinct $N = 2$ strings in general increases. In particular, for both the left and the right movers, one can still use the rotation symmetries on the fermions to bring $M + M^*$ to $\mathbb{I}$ but the symmetries acting X are reduced because there is less isometry. This affects the discussion of the space of inequivalent choices of $\mathcal{J}$ and $\tilde{\mathcal{J}}$. For all cases but the case of total compactification on a sufficiently irregular torus, the geometry is invariant under some parity changing rotation, so the $\mathbb{Z}_2$ symmetry that exchanges the two SU(2)'s always hold. Therefore the theory still has two broad types: α and β strings. We now consider the rest of the story case by case, using the stabilizers of subspaces of $\mathbb{R}^4$ given in section 2.1.4 and section 2.2.4. One notes that for a special size and shape of the torus the target space has nongeneric discrete isometries that reduce the parameter space of the $N = 2$ strings. For example, if a n-torus is rectangular, it will have ${\mathbb{Z}_2}^n$ inversion symmetries. If it is square it will also have power of $\mathbb{Z}_4$ symmetries. We shall not consider these situations here as our purpose here is not to exhaustively catalog all compactifications but rather to give the generic picture and to illustrate the systematic procedures. The extra symmetries can be easily taken into account by the same method and steps given here. A standard but obligatory caveat: when there is a compactified

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time-like direction it is not clear whether the physics of the theory degenerates. To be fair, one should first come up with a framework for physics in two times before such issues can even be meaningfully posed for N=2 string theories!

The rotational symmetry of a generic torus is a total inversion, which we denote here by $\mathfrak{I}$. So the parameter space of gauging is

$$\frac{(RP^2 \cup RP^{2\prime}) \times (RP^2 \cup RP^{2\prime})}{\mathfrak{I} \times \text{Isometry of the uncompactified space}} \tag{4.23}$$

for euclidean signature, and

$$\frac{(S^2_{1+} \cup S^2_{1+}{}') \times (S^2_{1+} \cup S^2_{1+}{}')}{\mathfrak{I} \times \text{Isometry of the uncompactified space}} \tag{4.24}$$

for $(2,2)$ signature. This applies for the generic case when there is a physically relevant and *generically* valued background for the B field on the torus. The CFT would then not have any symmetry that flips worldsheet parity. For special cases in which it does, the parameter space would instead be

$$\frac{(RP^2 \cup RP^{2\prime}) \times (RP^2 \cup RP^{2\prime})}{\mathbb{Z}_2 \times \mathfrak{I} \times \text{Isometry of the uncompactified space}} \tag{4.25}$$

for euclidean signature, and

$$\frac{(S^2_{1+} \cup S^2_{1+}{}') \times (S^2_{1+} \cup S^2_{1+}{}')}{\mathbb{Z}_2 \times \mathfrak{I} \times \text{Isometry of the uncompactified space}} \tag{4.26}$$

for $(2,2)$ signature. The $\mathbb{Z}_2$ is the symmetry that flips parity. In special cases where the background B vanishes or take special values so that the worldsheet theory is simply parity invariant, this $\mathbb{Z}_2$ just exchanges the left and right mover's space of choices. We can then evaluate the above one step further and obtain

$$\frac{\left[(RP^2 \times RP^{2\prime}) \cup \left(\frac{RP^2 \times RP^2}{\mathbb{Z}_2} \cup \frac{RP^{2\prime} \times RP^{2\prime}}{\mathbb{Z}_2}\right)\right]}{\mathfrak{I} \times \text{Isometry of the uncompactified space}} \tag{4.27}$$

for euclidean signature, and

$$\frac{\left[(S^2_{1+} \times S^2_{1+}{}') \cup \left(\frac{S^2_{1+} \times S^2_{1+}}{\mathbb{Z}_2} \cup \frac{S^2_{1+}{}' \times S^2_{1+}{}'}{\mathbb{Z}_2}\right)\right]}{\mathfrak{I} \times \text{Isometry of the uncompactified space}} \tag{4.28}$$

for $(2,2)$ signature.

4.4.1 $\mathbb{R}^3 \times S^1$

B field background is again irrelevant here. Therefore the worldsheet theory has parity symmetry which should be quotiented.

Euclidean space. In this case, the rotational isometry is reduced to O(3). This means $\mathrm{SU}(2)_O \times \mathrm{SU}(2)_{O'}$ is broken down to a diagonal SU(2) but the $\mathbb{Z}_2$ that exchanges them is not broken.

Therefore one can still bring $\mathcal{J}$ to the $\mathcal{J}^{[3]}$ for the left mover. The $\mathbb{Z}_2$ inversion of the S^1 does the same thing. In fact this inversion combined with an inversion of the R^3 does nothing to the complex structures.

For type β, it is the same situation as before, by a rotation that leave the left mover's $\mathcal{J}$ invariant one can reduce the choice of possibility to $\theta \in [0, \frac{1}{2}\pi]$. For type α, unlike in $\mathbb{R}^4$, we cannot rotate the $\tilde{\mathcal{J}}$ at will because only diagonal action of the two SU(2)'s are allowed now. Therefore the same situation as type β obtains: the theory is parameterized by $\theta \in [0, \frac{1}{2}\pi]$. Indeed, one can see the two cases more symmetrically. Because the two SU(2) are now coupled, one can think of the two complex structures for the left and right movers as two points on the *same* RP^2. An arbitrary rotation can fix one point to be, say, the north pole, then the residual rotation can fix the longitude of the other point. The remaining parameter is the latitude. Identifying antipodal points reduce the original S^2 to RP^2 and the range of latitude to $0, \ldots, \frac{1}{2}\pi$. The same picture applies to type α as well as β, even though for the same size of the circle, the two are distinct $N = 2$ strings. This is not a coincidence. As will be shown later, they are related by T-duality on the S^1.

In algebraic terms, the parameter space of gauging is calculated from eq. (4.27)

$$\frac{\left[(RP^2 \times RP^{2\prime}) \cup \left(\frac{RP^2 \times RP^2}{\mathbb{Z}_2} \cup \frac{RP^{2\prime} \times RP^{2\prime}}{\mathbb{Z}_2}\right)\right]}{\mathfrak{I} \times \mathrm{O}(3)} = \frac{RP^2 \times RP^{2\prime}}{\mathrm{SU}(2)} \cup \frac{RP^2 \times RP^2}{\mathrm{SU}(2)}\,. \tag{4.29}$$

Again the first part of the union corresponds to the α strings; the second β strings. Despite the presence of the prime on the α part, the two are actually identical since the SU(2) on the denominator acts simultaneously on both RP^2 and $RP^{2\prime}$ the same way.

(2, 2) space. In this case, the rotational isometry is reduced to O(2, 1). This means $\mathrm{SL}(2,\mathbb{R})_O \times \mathrm{SL}(2,\mathbb{R})_{O'}$ is broken down to a diagonal $\mathrm{SL}(2,\mathbb{R})$ but the $\mathbb{Z}_2$ that exchanges them remains unbroken. Therefore one can still bring $\mathcal{J}$ to the $\mathcal{J}^{[3]}$ for the left mover. For type β, it is the same situation as before, by a rotation that leaves the left mover's $\mathcal{J}$ invariant one can reduce the choice of possibility to $\gamma \in [0, \infty]$. For type α, unlike in $\mathbb{R}^4$, we cannot rotate the $\tilde{\mathcal{J}}$ at will because only diagonal actions of the two $\mathrm{SL}(2,\mathbb{R})$'s are allowed now. Therefore the same situation as type β obtains: the theory is parameterized by $\gamma \in [0, \infty]$. Indeed, one can see the two cases more symmetrically. Because the two $\mathrm{SL}(2,\mathbb{R})$ are now coupled, one can think of the two complex structures modulo signs for the left and right movers as two points on the *same* S^2_{1+}. By an $\mathrm{SL}(2,\mathbb{R})$ action one can fix one point to be at $r = 0$. This leaves an U(1) rotation that changes θ but does not affect r. It can fix the azimuthal angle of the other point. The remaining parameter is the r coordinate for the latter, which ranges from 0 to ∞. The same picture applies to type α as well as β, even though for the same size of the circle, the two are distinct $N = 2$ strings. This is not a coincidence. As will be shown later, they are related by T-duality on the S^1.

This discussion essentially parallels that of euclidean space but with the replacement of SU(2) by $\mathrm{SL}(2,\mathbb{R})$, S^2 by S^2_1, and RP^2 by S^2_{1+}. So the space of both α and β strings is a parameter r ranging from 0 to ∞. In algebraic terms this means that we have calculated from eq. (4.28) that

$$\frac{\left[(S^2_{1+}\times S^2_{1+}{}')\cup\left(\frac{S^2_{1+}\times S^2_{1+}}{\mathbb{Z}_2}\cup\frac{S^2_{1+}{}'\times S^2_{1+}{}'}{\mathbb{Z}_2}\right)\right]}{\mathfrak{I}\times \mathrm{O}(2,1)}=\frac{S^2_{1+}\times S^2_{1+}{}'}{\mathrm{SL}(2,\mathbb{R})}\cup\frac{S^2_{1+}\times S^2_{1+}}{\mathrm{SL}(2,\mathbb{R})}. \tag{4.30}$$

Again the first part of the union corresponds to the α strings; the second β strings. Despite the presence of the prime on the α part, the two are actually identical since the $\mathrm{SL}(2,\mathbb{R})$ on the denominator acts simultaneously on both S^2_{1+} and $S^2_{1+}{}'$ the same way.

4.4.2 $\mathbb{R}^2\times T^2$

Nonvanishing B field background on the Torus begins to play a role now. Unless $B=0$ or $B=\pi$ in some appropriate unit, worldsheet parity is broken, because it changes the sign of B. For nongeneric value or B and torus configuration, there is no symmetry in the CFT that changes the worldsheet parity. Left and right makes a difference and we generically should not quotient the parameter space by $\mathcal{J}\leftrightarrow\tilde{\mathcal{J}}$.

Euclidean space. In this case the rotational isometry is only $\mathrm{O}(2)$, it is embedded in the $\mathrm{O}(3)$ above as the stabilizer of an 3-vector. The parity changing elements in $\mathrm{O}(2)$ again ensure the dichotomy into α and β types as before. Also as before, we can visualize this as two points on the same RP^2. Breaking $\mathrm{O}(3)$ down to $\mathrm{O}(2)$ amounts to choosing a point on the S^2. The inversion $\mathfrak{I}$ inverts that point to its antipode and thus quotienting by it turns the point on S^2_1 into a point on S^2_{1+}. Let it be the north pole. The residual $\mathrm{U}(1)$ allows us to fix the longitude of one of the two points. The system is therefore naturally parameterized by two latitudes and one longitudes. The range of longitude is $0,\ldots,2\pi$, and that of latitude $0,\ldots,\pi$. This is the final answer if the two points are distinct. Since they correspond to $\mathcal{J}$ and $\tilde{\mathcal{J}}$ respectively, whether they are distinct or not depends on whether we have worldsheet parity symmetry. Generically we don't, as it would change the sign of the B field. If we do have parity symmetry because $B=0$ or $B=\pi$ in some appropriate unit, we have to further quotient the parameter space by the $\mathbb{Z}_2$ that exchanges $\mathcal{J}$ and $\tilde{\mathcal{J}}$. This means that we can assume a fixed ordering of for example the two latitudes, say $\theta_1\geq\theta_2$.

Algebraically, we have just evaluated eq. (4.23)

$$\frac{(RP^2\cup RP^{2\prime})\times(RP^2\cup RP^{2\prime})}{\mathfrak{I}\times \mathrm{O}(2)}=\frac{RP^2\times RP^{2\prime}}{\mathrm{U}(1)}\cup\frac{RP^2\times RP^2}{\mathrm{U}(1)} \tag{4.31}$$

for the generic case and eq. (4.27)

$$\frac{\left[(RP^2\times RP^{2\prime})\cup\left(\frac{RP^2\times RP^2}{\mathbb{Z}_2}\cup\frac{RP^{2\prime}\times RP^{2\prime}}{\mathbb{Z}_2}\right)\right]}{\mathbb{Z}_2\times\mathfrak{I}\times \mathrm{O}(2)}=\frac{RP^2\times RP^{2\prime}}{\mathbb{Z}_2\times \mathrm{U}(1)}\cup\frac{RP^2\times RP^2}{\mathbb{Z}_2\times \mathrm{U}(1)}, \tag{4.32}$$

for the parity invariant case. Again the first part of the union corresponds to the α strings; the second β strings. The story about the prime is also unchanged: despite it, α and β parts are identical.

As we have already mentioned several times and we will show in section 6, α and β strings are related by T-duality. We have just see this reflected in identical parameter space of gauging for α and β in S^1 compactification. The situation in T^2 is analogous but more subtle. The two parts are identical both for the generic case and for the parity invariant case. At first sight there is actually puzzling, because if we start with the nongeneric case of vanishing B on the torus, and suppose the torus is non rectangular, then T-dualizing along one of the periodically compactified direction will lead to another torus with nonvanishing B field. If this is naïvely thought as corresponding to the generic case one would have obtained a contradiction with T-duality as the parameter space of gauging would have differed by a factor of two between the two sides. However this thinking is incorrect. T-duality is an equivalence relation between different target space geometries and does not change the underlying CFT. In particular it cannot remove or create a symmetry. If the original theory is parity symmetric, its T-dual would also have it. However, since T-duality rotate the labels of the right moving fields, the worldsheet parity changing symmetry of the T-dual theory would have to involve a similar rotation between the labels of the left and right moving fields in addition to swapping them. With this taken into account, the parameter space of gauging of the T-dual theory is again eq. (4.32) instead of eq. (4.31) and there is no contradiction.

$(2,2)$ **space with spatial T^2.** In this case the rotational isometry is only $\mathrm{O}(2)$. It is embedded in the $\mathrm{O}(2,1)$ above as the stabilizer of a space-like 3-vector (i.e. associated with the (1) of (2,1)). The parity changing elements in $\mathrm{O}(2)$ again ensure the dichotomy into α and β types as before. Also as before, we can visualize this as two points on the same RP^2. Breaking $\mathrm{O}(2,1)$ down to $\mathrm{O}(2)$ amounts to choosing a point on the S^2_1. The inversion $\mathfrak{I}$ inverts that point to the other sheet and thus quotienting by it turns the point on S^2_1 into a point on S^2_{1+}. Let it be the point $r = 0$. The residual symmetry allows us to fix the azimuthal angle of one of the two points. The system is therefore naturally parameterized by one azimuthal angle and two radial parameters. The range of the former is $0, \ldots, 2\pi$, and that of later $0, \ldots, \infty$. This is the final answer if the two points are distinct. Since they correspond to $\mathcal{J}$ and $\tilde{\mathcal{J}}$ respectively, whether they are distinct or not depends on whether we have worldsheet parity symmetry. Generically we don't, as it would change the sign of the B field. If we do have parity symmetry for a particular configuration, we have to further quotient the parameter space by the $\mathbb{Z}_2$ that exchanges $\mathcal{J}$ and $\tilde{\mathcal{J}}$. This means that we can assume a fixed ordering of for example the two radial parameters, say $r_1 \geq r_2$.

Algebraically, we have just evaluated eq. (4.24)

$$\frac{(S^2_{1+} \cup S^2_{1+}{}') \times (S^2_{1+} \cup S^2_{1+}{}')}{\mathfrak{I} \times \mathrm{O}(2)} = \frac{S^2_{1+} \times S^2_{1+}{}'}{\mathrm{U}(1)} \cup \frac{S^2_{1+} \times S^2_{1+}}{\mathrm{U}(1)} \tag{4.33}$$

for the generic case and eq. (4.28)

$$\frac{\Big[(S^2_{1+} \times S^2_{1+}{}') \cup \Big(\frac{S^2_{1+} \times S^2_{1+}}{\mathbb{Z}_2} \cup \frac{S^2_{1+}{}' \times S^2_{1+}{}'}{\mathbb{Z}_2}\Big)\Big]}{\mathbb{Z}_2 \times \mathfrak{I} \times \mathrm{O}(2)} = \frac{S^2_{1+} \times S^2_{1+}{}'}{\mathbb{Z}_2 \times \mathrm{U}(1)} \cup \frac{S^2_{1+} \times S^2_{1+}}{\mathbb{Z}_2 \times \mathrm{U}(1)}. \tag{4.34}$$

for the parity invariant case. Again the first part of the union corresponds to the α strings; the second β strings. The story about the prime is also unchanged: despite it, α and β parts are identical. As the discussion about T-duality is the same we shall not repeat it.

$(2,2)$ space with one space and one time compactified. In this case the rotational isometry is only $\mathrm{O}(1,1)$. It is embedded in the $\mathrm{O}(2,1)$ above as the stabilizer of an 3-vector. The parity changing elements in $\mathrm{O}(2)$ again ensure the dichotomy into α and β types as before. We can again visualize this as two points on the same S^2_{1+}. Breaking $\mathrm{O}(2,1)$ down to $\mathrm{O}(1,1)$ amounts to choosing direction in the xy-plane and stratify S^2_1 into plane sections perpendicular to that direction and hence S^2_{1+}. Let us choose the direction be along y. The inversion $\mathfrak{I}$ is not relevant this time. The residual abelian symmetry acts transitively on the plane sections of S^2_{1+} as a Lorentz boost, with say z being the "time." This allows us to fix the x-coordinate of one of the two points to for example 0. The system is therefore naturally parameterized by one x and two y coordinates. There range would be all from $0,\ldots,\infty$. This is the final answer if the two points are distinct. Since they correspond to $\mathcal{J}$ and $\tilde{\mathcal{J}}$ respectively, whether they are distinct or not depends on whether we have worldsheet parity symmetry. Generically we don't, as it would change the sign of the B field. If we do have parity symmetry for a particular configuration, we have to further quotient the parameter space by the $\mathbb{Z}_2$ that exchanges $\mathcal{J}$ and $\tilde{\mathcal{J}}$. This means that we can assume that the unfixed x-coordinate is positive.

Algebraically, we have just evaluated eq. (4.24)

$$\frac{(S^2_{1+}\cup S^2_{1+}{}')\times(S^2_{1+}\cup S^2_{1+}{}')}{\mathfrak{I}\times\mathrm{O}(1,1)}=\frac{S^2_{1+}\times S^2_{1+}{}'}{\mathbb{R}}\cup\frac{S^2_{1+}\times S^2_{1+}}{\mathbb{R}} \tag{4.35}$$

for the generic case and eq. (4.28)

$$\frac{\left[(S^2_{1+}\times S^2_{1+}{}')\cup\left(\frac{S^2_{1+}\times S^2_{1+}}{\mathbb{Z}_2}\cup\frac{S^2_{1+}{}'\times S^2_{1+}{}'}{\mathbb{Z}_2}\right)\right]}{\mathbb{Z}_2\times\mathfrak{I}\times\mathrm{O}(1,1)}=\frac{S^2_{1+}\times S^2_{1+}{}'}{\mathbb{Z}_2\times\mathbb{R}}\cup\frac{S^2_{1+}\times S^2_{1+}}{\mathbb{Z}_2\times\mathbb{R}}. \tag{4.36}$$

for the parity invariant case. Again the first part of the union corresponds to the α strings; the second β strings. The story about the prime is also unchanged: despite it, α and β parts are identical. As the discussion about T-duality is the same we shall not repeat it.

4.4.3 $\mathbb{R}^1\times T^3$

Euclidean space. In this case the only "rotation" isometry of the noncompact space is is its $\mathbb{Z}_2$ inversion. The joint action of that inversion and $\mathfrak{I}$ has no effect on the complex structures. By themselves either can be used to show that the same α-β dichotomy obtains. The parameter space is that of two points on RP^2 with no symmetry to further reduce. Generically this is the final answer. For special value of the B field on the torus and suitable torus configuration, the theory can have a worldsheet parity changing symmetry and the parameter space of gauging becomes two identical points on RP^2. Algebraically, this gives

$$\left(RP^2\times RP^{2\prime}\right)\cup\left(RP^2\times RP^2\right) \tag{4.37}$$

for the generic case and

$$\frac{RP^2 \times RP^{2\prime}}{\mathbb{Z}_2} \cup \frac{RP^2 \times RP^2}{\mathbb{Z}_2} \tag{4.38}$$

for the parity symmetric case.

$(2,2)$ **space.** The same statement about the isometries for the euclidean space also applies here. The parameter space is that of two points on a S^2_{1+}. Generically the two points are distinct, but for cases where the CFT is parity invariant, they are identical.

4.4.4 T^4

In this rather special case, there is no compact directions left. Possible physical pathology aside, we can discuss the parameter space of gauging as it is. If the torus is the direct product of metric spaces in the form $S^1 \times T^3$ the answer is the same as the one for $\mathbb{R} \times T^3$ derived above because they have the same set of isometries.

If instead the T^4 is a generic one, the only isometry left is $\mathfrak{I}$, the total inversion of T^4. It has no effect on the complex structure. The algebraic result for the generic case is just eq. (4.16):

$$\left(RP^2 \times RP^{2\prime} \cup RP^{2\prime} \times RP^2\right) \cup \left(RP^2 \times RP^2 \cup RP^{2\prime} \times RP^{2\prime}\right). \tag{4.39}$$

for euclidean signature and and eq. (4.17):

$$\left(S^2_{1+} \times S^2_{1+}{}' \cup S^2_{1+}{}' \times S^2_{1+}\right) \cup \left(S^2_{1+} \times S^2_{1+} \cup S^2_{1+}{}' \times S^2_{1+}{}'\right). \tag{4.40}$$

for $(2,2)$ signature. When the $B=0$ or other special values such that the CFT is parity invariant, we should further reduce by $\mathbb{Z}_2$ and obtains

$$(RP^2 \times RP^{2\prime}) \cup \left(\frac{RP^2 \times RP^2}{\mathbb{Z}_2} \cup \frac{RP^{2\prime} \times RP^{2\prime}}{\mathbb{Z}_2}\right) \tag{4.41}$$

for euclidean signature and

$$(S^2_{1+} \times S^2_{1+}{}') \cup \left(\frac{S^2_{1+} \times S^2_{1+}}{\mathbb{Z}_2} \cup \frac{S^2_{1+}{}' \times S^2_{1+}{}'}{\mathbb{Z}_2}\right) \tag{4.42}$$

for $(2,2)$ signature. α string correspond to the first pair of parentheses and β the second. Note that there are disjoint subclasses with one or both types now. This is because we no longer have a target space parity changing symmetry to bring $\mathcal{J}$ to a particular S^2 or S^2_1. Put it in a geometric way, a generic T^4 does not treat the two classes of complex structures symmetrically.

Now eq. (4.41) and eq. (4.42) raise a puzzle similar to the one discussed on 34 and it has a similar resolution. The parameter space of gauging for α and β strings have the same "size," but are very different topologically spaces. In particular α string just has one component while β has two disjoint ones. There is no reason why the latter two should be joined as there is no continuous transition from one to the other. Does this contradict T-duality? Of course not. As we discussed in the last puzzle, the T-dual theory has a worldsheet parity changing symmetry that also involves relabeling the fields. That relabeling is itself like a parity operation on the target space. This is what happens with T-duality along a single direction and this is also precisely why it relates α and β strings: it swap the two S^2's or S^2_1's. Now quotient eq. (4.39) and eq. (4.4.4) by this $\mathbb{Z}_2$ and we get again eq. (4.41) and eq. (4.42) respectively, with one crucial difference: the parameter spaces for α and β are swapped. Therefore there is no contradiction with T-duality.

Note that a generic T^4 means all directions, include both "time" directions, are mixed with each other by the metric and periodicity conditions. While geometrically this is not unusual, its physical relevance and implication is far from obvious. But the reader has already been amply warned.

4.5 General dichotomy

We have shown that in all possible 4d flat space, $N = 2$ strings come in two broad flavors, α and β types, each in turn usually contains a continuous family. This seems very reminiscent of the dichotomy between type-IIA and -IIB superstrings. Indeed, as we shall see, when compactified on a circle, α and β $N = 2$ strings are T-dual to each other, much as type-IIA and -IIB superstrings. A question then naturally arises as to whether this dichotomy is special to flat space or can be generalized to other spacetime geometry. Here we give an affirmative answer.

Supersymmetric σ-models can be conveniently written in terms of $N = 1$ superspace:[14]

$$S = \int d^2\sigma d^2\theta \, (\mathcal{G}_{IJ} + \mathcal{B}_{IJ}) \mathbf{D}\mathbf{X}^I \tilde{D}\mathbf{X}^J \,. \tag{4.43}$$

As written, it is already supersymmetric: there is no condition on the geometric quantity $\mathcal{G}$ and $\mathcal{B}$. For this reason, it is believed that all σ-models with $N = 1$ supersymmetry can be described by eq. (4.43), including those with extended supersymmetry, as extended supersymmetry algebra always include the $N = 1$ supersymmetry algebra as a proper subalgebra. Therefore the following discussion and results, in particular the appearance of two almost complex structures, are not made less general by the use of $N = 1$ superspace, which is just for convenience. In [17], $N = 2$ (as well as $N = 4$) σ-models were considered in this framework by introducing a second pair of supersymmetry transformations defined by[15]

$$\mathbf{Q}_2 \mathbf{X}^I = \mathcal{J}^I{}_J \mathbf{D}\mathbf{X}^J \,, \qquad \tilde{\mathbf{Q}}_2 \mathbf{X}^I = \tilde{\mathcal{J}}^I{}_J \tilde{\mathbf{D}}\mathbf{X}^J \,. \tag{4.44}$$

They derived from $N = 2$ supersymmetry algebra that both $\mathcal{J}$ and $\tilde{\mathcal{J}}$ are integrable complex structures. They also found that the invariance of the action eq. (4.43) under the second supersymmetry implies that both complex structures make $\mathcal{G}$ hermitian and that they are respectively covariantly constant:

$$D\mathcal{J} = \tilde{D}\tilde{\mathcal{J}} = 0 \,. \tag{4.45}$$

[14]For a brief review of $N = 1$ superspace in 2d, see, for example, section 4.1.2 of [16].

[15]We note that the authors of [17] considered compact $N = 2$ supersymmetry, in sense that the R-symmetry group is a compact U(1). That is precisely the case relevant to this paper.

The caveat is that D and $\tilde{D}$ are two covariant derivatives defined respectively with two affine connections given by

$$\Gamma = \Gamma_0 + \mathcal{H}\,, \qquad \tilde{\Gamma} = \Gamma_0 - \mathcal{H}\,, \tag{4.46}$$

where all (implicit) indices are downstairs and Γ_0 is the Levi-Civita connection associated with $\mathcal{G}$. From this they constrains $\mathcal{J}$ and $\tilde{\mathcal{J}}$ and determine $\mathcal{H}$ in terms of them.

Now for a given σ-model, there can be no, one or more than one candidate for $\mathcal{J}$ and $\tilde{\mathcal{J}}$, depending on the number of extended supersymmetry the theory has. As a pre-theory for $N = 2$ string, we require there to be at least one candidate for $\mathcal{J}$ and one for $\tilde{\mathcal{J}}$. Though the simplest and most widely studied cases have $\mathcal{J} = \tilde{\mathcal{J}}$, it does not have to be so [17]. There can also be more than one choices for $\mathcal{J}$ and $\tilde{\mathcal{J}}$. If so the extended supersymmetry is more than $N = 2$. The choice is irrelevant for the σ-model itself, other than that their existence implies the existence of a bigger symmetry group and $N = 2$ supersymmetry and U(1) R-symmetry. However, for the $N = 2$ string the σ-model is only the pre-theory. To specify the string theory, we have to make a specific choice for $\mathcal{J}$ and $\tilde{\mathcal{J}}$ respectively. This choice determines the action of the U(1) R-symmetry and hence the left and right affine U(1) currents of the $N = 2$ SCA to be gauged. Different choices lead to different $N = 2$ string theory. In section 4 we have seen this to happen when the target space is flat, and we have examined the space of physically inequivalent choices. We shall return to examine cases of curved target spaces in section 5.

This choice allows us to generalize the α and β dichotomy to arbitrary spacetime $\mathfrak{X}$ described by $N = 2$ σ-models. As shown in section 2.1.3 and section 2.2.3, the space of complex structures at each point in a $4d$ space is the disjoint union of two identical manifolds. For euclidean metric, it is two S^2; for $(2, 2)$ metric, it is two S^2_1. A complex structure for the whole manifold is a section of a fibration of this disjoint union over $\mathfrak{X}$. This fibration is completely determined by the topology of the target space manifold, as the transition function for the fibers can be related to that of the tangent bundle, which is either O(4) or O(2, 2) depending on the signature of the metric. Special orthogonal elements acts on the two component manifolds separately; elements that change the orientation of $\mathbb{R}^4$ map one to the other S^2 or S^2_1. While locally on each coordinate patch one can distinguish the two manifolds, whether this distinction can be extended globally over the whole target space is a different matter.

Abstractly speaking this labeling yields a vector bundle over the target space whose fiber is the vector space $\mathbb{Z}_2$ understood as a module of the abelian group of $\mathbb{Z}_2$. Its transition function between two patches is the sign of the determinant of the corresponding transition function of the tangent bundle. This $\mathbb{Z}_2$ bundle is trivial if and only if the first Stiefel-Whitney class of $\mathfrak{X}$ vanishes, i.e. $\mathfrak{X}$ is orientable. On the other hand, nonorientability is an obstruction to the existence of an almost complex structure because parity changing maps are not holomorphic. For example, on a 2d surface reflection along the $dz + d\bar{z}$ direction exchanges dz and $d\bar{z}$. Any other reflection (parity) along a certain direction does the same with some additional phase factor. Therefore one cannot globally distinguish dz and $d\bar{z}$, which is what an almost complex structure does, if one cannot eliminate all reflections from the transition functions of tangent bundles across patches, which is what nonorientability means. This argument easily generalizes to higher dimensions. Therefore an almost complex manifold is necessarily orientable, and so we should restrict ourselves to this case since $\mathfrak{X}$ is almost complex with $\mathcal{J}$ and $\tilde{\mathcal{J}}$.

On orientable $\mathfrak{X}$, the two locally disjoint components of the fiber are also globally distinct. The fibration therefore also splits into two components. Each one is a fibration of S^2 or S^2_1 over the target space. In defining the σ-modeleq. (4.43), we have to make a choice for $\mathcal{J}$ and another for $\tilde{\mathcal{J}}$. There are in general many choices. The first level of classification is given by the discrete choices of which components of the fibration they respectively belong.

> *We define $N = 2$ α string as those defined with $\mathcal{J}$ and $\tilde{\mathcal{J}}$ from different components. $N = 2$ β strings, on the other hand, have them from the same component.*

5. $N = 2$ strings in curved spaces

Flat space is not the only space in which $N = 2$ string can propagate. Since any $N = 2$ SCFT with the appropriate central charge is a starting point for constructing a $N = 2$ string theory, it is interesting and imperative to investigate the case of interacting $N = 2$ SCFTs. Some of them have spacetime interpretation, being $N = 2$ superconformal invariant σ-models. These have been studied intensively, mainly because $N = 2$ SCFT provides supersymmetric compactification for the superstring theories. They can be used for $N = 2$ string if their central charge is 6, which usually translates to 4 real dimensions. To our knowledge, reasonably rigorous analysis that completely classifies or characterizes such spaces without extra (hidden) assumption does not yet exist. Instead, there are fairly large class of examples characterized by some additional geometric conditions such as holonomy, torsion, and superspace representations. In this section, we shall consider several of them. The aim is to solve for some families of $N = 2$ string theories one can define in these spaces. The principles of such analysis have already been laid down in section 4.1.2. We restrict ourselves to simple embeddings of $N = 2$ SCAs in the $N = 4$ SCA, to defined shortly, that is characteristic of all the examples we shall study. We first establish a general result about all such embeddings. Then we remove the degeneracy among them caused by the symmetry of the corresponding field theory. There are interesting examples of curved spaces in which α string propagate, but here we concentrate on β strings. Unlike the case of flat space, they generally do not coexist in the same curved spacetime. This does not, however, mean that they are unrelated, but we shall wait until section 6 to see the relation between them. Both $(4, 0)$ and $(2, 2)$ signatures are considered.

The recipe in section 4.1.2 requires us to find out all such symmetries given that we have all the information about the geometry of $\mathfrak{X}$. If we also have enough control over quantum corrections, we can also find out how at the quantum level they act on the realizations of the $N = 2$ SCA's we want to gauge. Sometimes, however, we are just given with some general features of the pre-theory, such as the presence of a bigger symmetry algebra $\mathfrak{A}$. For the examples in this section $\mathfrak{A}$ is the $N = 4$ SCA. We then have to make the assumption of genericness that this is all the symmetry the theory has, which is already is quite informative. In lieu of the assumption of simple realization in terms of fundamental matter fields eq. (4.1), we solve for all "simple" embeddings of the $N = 2$ SCA in $N = 4$ SCA. They are related to the automorphisms of group of $\mathfrak{A}$, the detail depends on $\mathfrak{A}$. That has to be quotiented by the symmetry of the theory, which at least include the inner automorphisms of $N = 4$ SCA. For other chiral algebra $\mathfrak{A}$, the procedure would be analogous.

5.1 Simple embeddings of $N = 2$ SCA in $N = 4$ SCA

5.1.1 Compact $N = 4$

Define a simple embeddings of a compact $N = 2$ SCA in a compact $N = 4$ SCA by

$$G^+ = m_{\alpha\beta}G^{\alpha,\beta}, \qquad G^- = -m^*_{\alpha\beta}(\sigma^2)^\alpha{}_\gamma(\sigma^2)^\beta{}_\lambda G^{\gamma,\lambda}\,. \tag{5.1}$$

We can write m in terms of Pauli's σ matrices (see the appendix for our conventions):

$$m_{\alpha\beta} = (m_0\epsilon + \imath m_i\sigma^i)_{\alpha\beta}\,. \tag{5.2}$$

Note that the $\mathrm{SU}(2)_O \times \mathrm{SU}(2)_I$ symmetry acts on m by rotating m^I as a 4-vector under $\mathrm{SO}(4)$. We want $G^\pm$ has the same OPE as eq. (3.1), with J given by

$$J = 2a_iJ^i\,. \tag{5.3}$$

This problem is similar to the one we have solved in section 4.2. By a similar analysis, we find that

$$m_I = \frac{1}{2}(p_I + \imath q_I), \qquad p_I, q_I \in \mathbb{R} \tag{5.4}$$

such that

$$\langle p,p\rangle = \langle q,q\rangle = 1\,, \qquad \langle p,q\rangle = 0\,, \tag{5.5}$$

and

$$-q_0p_k + p_0q_k + \epsilon^{ijk}p_iq_j = a_k\,. \tag{5.6}$$

We could continue to solve these and other equations resulting from requiring eq. (3.1). Instead, let us note that given eq. (5.5), we can, by an $\mathrm{SO}(4)$ rotation, bring p and q to a preferred form:

$$\begin{aligned} &p_I = 0\,, \quad I = 1,2,3\,; \qquad p_2 = 1;\\ &q_I = 0\,, \quad I = 0,1,3\,; \qquad q_1 = -1\,. \end{aligned} \tag{5.7}$$

Then,

$$a_3 = 1\,, \qquad a_1 = a_2 = 0\,, \tag{5.8}$$

and

$$G^+ = G^{1,1}\,, \qquad G^- = G^{2,2}\,, \qquad J = 2J^3\,. \tag{5.9}$$

Thus we have proved that every simple embeddings of $N = 2$ SCA is related to eq. (3.16) by some action of $\mathrm{SU}(2)_O \times \mathrm{SU}(2)_I$ automorphism.

Hence in order to find the parameter space of gauging we should start with $\mathrm{SU}(2)_O \times \mathrm{SU}(2)_I$. Right away we can drop $\mathrm{SU}(2)_I$ because, as internal automorphism, they are necessarily symmetry of the CFT. Furthermore, a U(1) subgroup of $\mathrm{SU}(2)_O$ just changes the phase $G^{\pm}$ in opposite directions. They do not give different embeddings and should also be quotiented out. Therefore we get $\mathrm{SU}(2)/\mathrm{U}(1) \simeq S^2$. Finally, there is a $\mathbb{Z}_2$ operation in $\mathrm{SU}(2)_I \times \mathrm{SU}(2)_O$ that reverses the sign of J and interchanges $G^{1,1}$ and $G^{2,2}$. This $\mathbb{Z}_2$ is the Weyl group in each SU(2) and it also leads to the same $N = 2$ SCA. Indeed it corresponds to the conjugation automorphism of the latter. We are now left with

$$\frac{S^2}{Z_2} = RP^2 \,. \tag{5.10}$$

At least part of the symmetries of the theory have already been taken into account to arrive at this result.

5.1.2 Noncompact $N = 4$

Define a simple embeddings of a compact $N = 2$ SCA in a noncompact $N = 4$ SCA by

$$G^+ = m_{\alpha\beta} G^{\alpha,\beta}\,, \qquad G^+ = m^*_{\alpha\beta} G^{\alpha,\beta}\,. \tag{5.11}$$

We can write m in terms of sigma matrices (see the appendix for our notations):

$$m_{\alpha\beta} = (m_0 \epsilon + \imath m_2 \sigma^2 + m_1 \sigma^1 + m_3 \sigma^3)_{\alpha\beta}\,. \tag{5.12}$$

Note that the $\mathrm{SL}(2,\mathbb{R})_O \times \mathrm{SL}(2,\mathbb{R})_I$ symmetry acts on m by rotating m^I as a 4-vector under $\mathrm{SO}_0(2,2)$. We want $G^{\pm}$ has the same OPE as eq. (3.1), with J given by

$$J = 2 a_i J^i \,. \tag{5.13}$$

This problem is similar to the one we have solved in section 4.2. By a similar analysis, we find that

$$m_I = \frac{1}{2}(p_I + \imath q_I), \qquad p_I, q_I \in \mathbb{R} \tag{5.14}$$

such that

$$\langle p,p \rangle = \langle q,q \rangle = 1\,, \qquad \langle p,q \rangle = 0\,, \tag{5.15}$$

and

$$q_0 p_k - p_0 q_k + \varepsilon_k^{ij} p_i q_j = a_k\,. \tag{5.16}$$

We could continue to solve these and other equations resulting from requiring eq. (3.1). Instead, let us note that given eq. (5.15), we can, by an $\mathrm{SO}_0(2,2)$ rotation, bring p and q to a preferred form:

$$\begin{aligned} p_I &= 0\,, \quad I = 1,2,3\,; \qquad p_0 = 1\,; \\ q_I &= 0\,, \quad I = 0,1,3\,; \qquad q_2 = \mp 1\,. \end{aligned} \tag{5.17}$$

Then,

$$a_2 = \pm 1\,, \qquad a_1 = a_3 = 0\,. \tag{5.18}$$

The choice of the sign cannot yet be fixed because flipping it while keep $p_0 = 1$ requires the use of an improper element of SO(2,2). However it does not matter, because flipping that sign correspond to reversing J and interchanging $G^{\pm}$. That does not give us a different $N = 2$ SCA. So we make a choice and find that $G^{\pm}$ and J are given as in eq. (3.24) and eq. (3.25) respectively. Thus we have proved that every simple embeddings of $N = 2$ SCA is related to eq. (3.24) by some action of $\mathrm{SL}(2,\mathbb{R})_O \times \mathrm{SL}(2,\mathbb{R})_I$ automorphism.

Hence in order to solve for the parameter space of gauging we should start with $\mathrm{SL}(2,\mathbb{R})_O \times \mathrm{SL}(2,\mathbb{R})_I$ Right away we can drop $\mathrm{SU}(2)_I$ because, as internal automorphism, they are necessarily symmetry of the CFT. Furthermore, a U(1) subgroup of $\mathrm{SL}(2,\mathbb{R})_O$ just changes the phase $G^{\pm}$ in opposite directions. It leads to the same $N = 2$ SCA and thus should be quotiented out. We are now left with

$$\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1) \simeq S^2_1\,. \tag{5.19}$$

Finally, there is a $\mathbb{Z}_2$ operation on $\mathrm{SL}(2,\mathbb{R})_I \times \mathrm{SL}(2,\mathbb{R})_O$ that reverses the sign of J and interchanges $G^{1,1}$ and $G^{2,2}$. This $\mathbb{Z}_2$ is the Weyl group in each $\mathrm{SL}(2,\mathbb{R})$ and the same as the improper element of SO(2,2) mentioned above. It also leads to the same $N = 2$ SCA. Indeed it corresponds to the conjugation automorphism. We are now left with

$$\frac{S^2_1}{Z_2} = S^2_{1+}\,, \tag{5.20}$$

At least part of the symmetries of the theory have already been taken into account to arrive at this result.

5.1.3 $N = 2$ in fat, free $N = 4$, a second look

Before going into the curved spaces we shall do a warm-up exercise by applying the results above to the case of flat space and recover the same results as obtained there from free fields. In fact, this subsection and section 4.2 can be read independently.

The various automorphism and symmetry of the fat, free $N = 4$ SCA discussed in section 3.3 are clearly related to the symmetry of $\mathbb{R}^4$, discussed in detail in section 2. Now consider a σ-model defined on this $\mathbb{R}^4$, the chiral algebra of which has been discussed in section 3.3. The rotational symmetry can act on the bosons X^I's and fermions Ψ^I's. Because the bosons and fermions decouple from each other, one can in fact rotate them independently. Furthermore, because the left and right moving chiral fermions are actually different fields, one can also rotate them further. This means that in fact there are left and right chiral affine SU(2) or $\mathrm{SL}(2,\mathbb{R})$ currents corresponding to them. They are the J^i, J'^i, and their right chiral counterparts $\tilde{J}^i$ and $\tilde{J}'^i$, in eq. (3.29) or eq. (3.33). The bosons, however, cannot be completely separated into independent left and right moving parts. The space being noncompact means that the left and right movers share the same zero modes. Therefore rotation of the bosons leads to just one rotation symmetry group that rotates the X, which does not correspond to any chiral current. The same story applies to parity operations that swaps the two spinor indices. For fermions each leads to two symmetries, one for each chirality. For bosons there is just one acting on X.

Even though there is no symmetry rotating the left and right moving parts of X independently, the chiral algebra eq. (3.26) *is* invariant under independent rotation of ∂X and $\bar{\partial} X$. This is an example where an automorphism of the chiral algebra does not correspond to a symmetry of the theory. Therefore it is necessarily an outer automorphism. The origin of all the symmetry and automorphism of the fat, free $N = 4$ SCA should be evident now.

Given the structure of the fat, free $N = 4$ SCA, we learned from section 3.3 that there are at least 4 distinct ways to embed an $N = 4$ SCA, depending on which of the fours quadruplets of supercharges, G_{00}, G_{01}, G_{10}, and G_{11}, one choose. By a parity operation on Ψ, these choices can be narrowed down between G_{00} and G_{10}. After having chosen which way to embed the $N = 4$ SCA, there remain a continuum of choices to finally fix the $N = 2$ SCA embedding. As discussed section 3.2, if the affine group is SU(2), the space of different simple embeddings mod $\mathrm{SU}(2)_I$ is RP^2; if the affine group is $\mathrm{SL}(2, \mathbb{R})$ instead, the space of embeddings mod $\mathrm{SL}(2, \mathbb{R})$ is S^2_{1+}. We have seen in section 3.3 that these two possibilities correspond to $(4, 0)$ and $(2, 2)$ metrics respectively. Now putting the left and right movers together and we have

$$((RP^2 \cup RP^{2\prime}) \times (RP^2 \cup RP^{2\prime})) \tag{5.21}$$

and

$$((S^2_{1+} \cup {S^2_{1+}}') \times (S^2_{1+} \cup {S^2_{1+}}')) \tag{5.22}$$

for the two types of signatures respectively. They are identical to eq. (4.16) and eq. (4.17). One can then continue with the analysis of flat spaces in section 4.3 and section 4.4. We remark that when the left and right chiral $N = 2$ supercurrents come from G_{00} and $\tilde{G}_{10}$ respectively, we have an α-string. When the left and right chiral $N = 2$ supercurrents both come from G_{00}, we have a β-string.

5.2 β-string in curved space

As defined in section 4.5, $N = 2$ β string is describable by a σ model with $N = 2$ such that the left and right movers use the same complex structure.

K3. A generic $K3$ has no isometry. Because the worldsheet field theory is generically not free, we cannot apply the analysis of 3.2. That would of course be possible for orbifold limits of $K3$, but here we restrict ourselves to the generic case. Fortunately, it is believed that the supersymmetric σ-model with $K3$ as target space is in fact a $N = 4$ SCFT [18]. For the generic case, the only symmetries of that theory are those generated by the $N = 4$ SCA. So we content ourselves with simple embedding $N = 2$ SCA in the $N = 4$ SCA as defined in eq. (5.1) and directly use the result thereof, i.e. the space of gauging parameters is

$$(RP^2 \times RP^2) \tag{5.23}$$

with no further reduction. Here we are taking account of both the left and right movers. This space represents two distinct points on RP^2 and is applicable for a generic value of the B field. For special values of the B field the worldsheet theory also preserve worldsheet parity and we should quotient eq. (5.23) by the $\mathbb{Z}_2$ that exchanges the two RP^2. It represents two identical points of RP^2. This result applies not just to σ-model on generic $K3$, but also to any $N = 4$ SCFT background for $N = 2$ β string that has no additional symmetries.

Isometry. What about a target space with isometry? The isometries lead to additional symmetries of the theory that involve the coordinate fields X^I. While some of them may commute with the $N = 4$ SCA's, if there is any that does not, it will relate two different embeddings of $N = 2$ SCA. It cannot be an inner automorphism of the $N = 4$ SCA, because the latter only affect the fermions. So they have to quotiented out from the result obtained above. One possible scenario would be that the target space has an SU(2) isometry, which corresponds to the a diagonal action of the left and right $N = 4$ SCA's SU(2) outer-automorphisms. This would reduce the space of choices to just one RP^2. This result holds whether or not the CFT is parity invariant because there are two elements of SU(2) that acts the the same way as the $\mathbb{Z}_2$ exchange symmetry: exchanging $\mathcal{J}$ and $\tilde{\mathcal{J}}$.

Examples of hyper-Kähler manifolds with continuous isometries can be found in [19]. For example both the ALE spaces and multi-Taub-NUT spaces have self-dual Riemann tensor and have a large isometry group that is $\mathrm{SU}(2) \times \mathrm{U}(1)$. To carries out the above procedure one needs to identity the algebra of the isometry induced symmetries of the field theory and the $N = (4, 4)$ supercharges. This is currently under study.

$(2, 2)$ **spaces.** Curved pseudo-hyper-Kähler spaces, i.e. one with metric of $(2, 2)$ signature leading to a self-dual Riemann tensor, is also worth considering. In this paper we content ourselves with outlining the general procedures rather than giving specific examples. If such a space has no isometry and the holonomy group is not further reducible, the corresponding SCFT is likely to have no symmetry other than $N = 4$ SCA. We can then carry over the analysis for a generic $K3$ and replacing SU(2) by $\mathrm{SL}(2, \mathbb{R})$ in the discussion. Again, we will content ourselves with simple embedding compact $N = 2$ SCA in the noncompact $N = 4$ SCA. The space of gauging parameters is thus

$$(S^2_{1+} \times S^2_{1+})\,. \tag{5.24}$$

Here we are taking account of both the left and right movers. When the CFT has parity symmetry, we should also quotient by it. The result is

$$(S^2_{1+} \times S^2_{1+})/\mathbb{Z}_2\,. \tag{5.25}$$

If there is isometry, we have to mod out its action as well.

6. T-duality and $\alpha \leftrightarrow \beta$ conversion

In this section we show as promised that $N = 2$ α and β strings are related by T-duality.

6.1 Flat space

T-duality and its effect on the target space geometry is well-known. Here we are interested in its relevance to $N = 2$ strings. First we note that as T-duality is a relabeling of fields that relates one CFT to two different looking σ-models, it obviously has to exist for toroidally compactified $N = 2$ strings. It can be thought as the following field redefinition[16]

$$\begin{aligned} \partial X'^I &= \partial X^I\,, & \bar\partial X'^I &= \mathcal{R}^I{}_J \bar\partial X^J\,; \\ \Psi'^I &= \Psi^I\,, & \bar\Psi'^I &= \mathcal{R}^I{}_J \bar\Psi^J\,. \end{aligned} \tag{6.1}$$

For consistency with the Virasoro algebra, the OPE of $\bar\partial X' \bar\partial X'$ and $\bar\Psi'\bar\Psi'$ must not change. Hence $\mathcal{R}$ must be an element of the orthogonal group $\mathrm{O}(4)$ or $\mathrm{O}(2,2)$ [26]. If we want to keep the same $N = 2$ string theory, the (super)currents we gauge must not change. However, $\tilde J$ now has a different expression in terms $\bar\partial X'$ and $\bar\Psi'$ while J retains the same form:

$$\tilde J = \frac{\imath}{2} \tilde{\mathcal{J}}^I_J : \Psi_I \Psi^J := \frac{\imath}{2} \tilde{\mathcal{J}}'^I_J : \Psi'_I \Psi'^J : \tag{6.2}$$

where $\tilde{\mathcal{J}}'$ is related to $\tilde{\mathcal{J}}$ by the rotation $\mathcal{R}$:

$$\tilde{\mathcal{J}}' = \mathcal{R}^\top \tilde{\mathcal{J}} \mathcal{R}\,. \tag{6.3}$$

$\mathcal{R}$ is further constrained by modular invariance that is determined by the charge lattice of momenta and winding numbers, but for any toroidal compactification it is nontrivial and in particular contain elements with determinant -1.

Such elements are of special interest to $N = 2$ strings, because as we have discussed in section 2.1.3 and section 2.2.3, they map complex structure from one S^2 or S^2_1 to the other. Therefore they exchange the α and β types of $N = 2$ strings. The simplest example is compactification of X^4 on S^1. With

$$\mathcal{R} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{6.4}$$

representing the T-duality on the S^1, $\tilde{\mathcal{J}}^{[3]}$ is mapped to $tilde\mathcal{J}'^{[3]}$. With more work, one can work out how the parameters for α and β strings are mapped into each other for all toroidal compactification and all T-duality operations.

6.2 Curved space

Flat space is of course very special. We would like to know if the T-duality relation between α and β strings also holds for curved spaces. What is needed is a version of T-duality that also works for curved spaces. Fortunately, this had already been studied in the literature and it is worthwhile to invoke and reexamine these known results in the context of the

[16] For a quick look at the facts of T-duality used here, see for example lecture two of [20]. For a comprehensive review of of T-duality, see for example [21].

spacetime understanding we have developed in the this work. They are couched in the language of $N = 2$ superspace. This formalism provides a compact and efficient way to write down an $N = 2$ σ-model, although it is not yet known that it is versatile enough to cover all possible cases.

We shall not repeat here the details of $N = (2, 2)$ superspace in 2d. See for example [17] for a summary. It has two copies of the same superalgebras, one for the left and one for the right movers. Denote the holomorphic super-derivatives for the left and right movers by D and $\tilde{D}$ respectively. The antii-homolomprphic ones are $\bar{D}$ and $\tilde{\bar{D}}$. Since $\{D, \tilde{D}\} = \{D, \tilde{\bar{D}}\} = 0$, in addition to the (anti-)chiral superfield $\Phi(\bar{\Phi})$[17]

$$\bar{D}\Phi = 0\,, \qquad \tilde{\bar{D}}\Phi = 0\,, \tag{6.5}$$

we have the twisted (anti-)chiral superfield $X(\bar{X})$ that satisfies

$$\bar{D}X = 0\,, \qquad \tilde{D}X = 0\,. \tag{6.6}$$

It is easy to see that the T-duality along, say, $\Phi + \bar{\Phi}$, turns a chiral superfield into a twisted chiral superfields and an anti-chiral superfield into a twisted anti-chiral superfields [22]. What this means in terms of target space geometry is that, relative to the left mover, the complex structure of the right mover changes sign, so the notion of (anti-)chirality is inverted for the superalgebra of the right mover. Now put it in the context of a $N = 2$ string theory. The critical dimension is 4. Let us start with β theory, with both complex structures being, with O(4) vector indices,

$$\mathcal{J} = \begin{pmatrix} \imath\sigma^2 & 0 \\ 0 & \imath\sigma^2 \end{pmatrix}. \tag{6.7}$$

So we have two pairs of chiral and anti-chiral multiplets. After a T-duality transformation along, say, the 4th coordinate, we would instead find one chiral and anti-chiral pair plus one twisted and anti-twisted pairs. Therefore the T-duality affects the complex structure. The left mover should have the same complex structure for as before, $\mathcal{J}$, since T-duality does not affect the left movers. Therefore the complex structure for the right mover is now

$$\tilde{\mathcal{J}} = \begin{pmatrix} \imath\sigma^2 & 0 \\ 0 & -\imath\sigma^2 \end{pmatrix}. \tag{6.8}$$

It is easily verified that $\mathcal{J}$ and $\tilde{\mathcal{J}}$, which commute, resides on the two different sphere of complex structures respectively. In the above we have used complex structure for euclidean space, but it is easy to verify that the same applies to $(2, 2)$ spaces.

It is a general fact in 4d that if two different complex structures commute with each other, they belong to two different spheres or hyperboloids of complex structures. This following from (eqs. 2.23), (2.24), (2.54), (2.54)). In the above example, β string with identical complex structures for the left and right movers is turned into α strings. However, β string can be more general in that the two complex structures may be different while

[17]We shall not need their component forms, which are of course expansions in powers of two real Grassmann coordinates.

lying on the same sphere or hyperboloid. It follows from the same set of equations in section 2 that they do not commute. Indeed it is clear that, if $\mathcal{J}$ and $\tilde{\mathcal{J}}$ are on the same sphere/hyperboloid but $\mathcal{J} \neq \tilde{\mathcal{J}}$, $[\mathcal{J}, \tilde{\mathcal{J}}]$ is non-degenerate. Such cases, to the extent an $N = 2$ superspace representation is known, use exactly one of what has been called a semi-chiral multiplet [23]. It is also known that a semi-chiral multiplet is related to one (anti-)chiral and one (anti-)twisted chiral multiplet by a T-duality [24]. This completes the picture of the T-duality relation between α and β strings whenever a $N = 2$ superspace representation is known.

The above discussion may seem somewhat abstract. Now we draw from the literature [22] a concrete example of how this T-duality would affect the $N = 2$ σ-model formulate in $N = 2$ superspace. Consider the β theory and the action for an $N = 2$ σ-model

$$S_B = \int d^2x d^2\theta d^2\bar{\theta} K(\phi^1, \phi^2, \bar{\phi}^1, \bar{\phi}^2)\,, \tag{6.9}$$

where $\phi^i, i = 1, 2$ are two chiral superfields. The real function K is called the Kähler potential and the Kähler metric is given by

$$G_{i\bar{j}} = \frac{\partial^2 K}{\partial\phi^i \partial\bar{\phi}^{\bar{j}}}\,. \tag{6.10}$$

If we gauge the U(1) R symmetry chosen by this representation we would have a β string with $\mathcal{J} = \tilde{\mathcal{J}}$.

Suppose now that the metric eq. (6.10) admits an action of a holomorphic isometry. Then there exists a local holomorphic coordinate system in which

$$K(\phi^1, \phi^2, \bar{\phi}^1, \bar{\phi}^2) = K(\phi^1 + \bar{\phi}^1, \phi^2, \bar{\phi}^2)\,, \tag{6.11}$$

with the isometry generated locally by $\partial/\partial(\phi^1 - \bar{\phi}^1)$. We can write the action eq. (6.9) as

$$S = \int d^2\sigma d^2\theta d^2\bar{\theta}\, \left(K(V, \phi^2, \bar{\phi}^2) - (\chi + \bar{\chi})V\right), \tag{6.12}$$

where V is an unconstrained real superfield and χ is a twisted chiral superfield. Solving the field equations for χ and $\bar{\chi}$ yields $V = \phi^1 + \bar{\phi}^1$, which brings us back to the action (eq. (6.9)) with the Kähler potential eq. (6.11). On the other hand, solving the field equation for V yields a dual action

$$\hat{S} = \int d^2\sigma d^2\theta d^2\bar{\theta}\, \left(\hat{K}(\chi + \bar{\chi}, \phi^2, \bar{\phi}^2)\right), \tag{6.13}$$

where $\hat{K}$ is the Legendre transform of K with respect to $\phi^1 + \bar{\phi}^1$. The function $\hat{K}$ encodes the geometry of the new target space. The new metric $\hat{G}$ is obtained from $\hat{K}$ via

$$\hat{G}_{1\bar{1}} = -\frac{\partial^2 \hat{K}}{\partial\chi\partial\bar{\chi}}\,, \qquad \hat{G}_{2\bar{2}} = \frac{\partial^2 \hat{K}}{\partial\phi^2\partial\bar{\phi}^2}\,, \tag{6.14}$$

and the other components are zero. The new background is conformally invariant at one-loop. It has a B field

$$B_{1\bar{2}} = \frac{\partial^2 \hat{K}}{\partial\chi\partial\bar{\phi}^2}\,, \qquad B_{2\bar{1}} = \frac{\partial^2 \hat{K}}{\partial\phi^2\partial\bar{\chi}}\,, \tag{6.15}$$

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and the other components are zero. In addition there is a dilaton φ given by

$$\varphi = \frac{1}{2} \log \hat{G}_{1\bar{1}} \,. \tag{6.16}$$

The geometry of this space has been studied in [17]. It is a hermitian space which is locally a product. It has two commuting complex structures $J^{\pm i}_{j}$. They are covariantly constant with respect to to a connection with torsion $H = dB$, $\Gamma^{\pm a}_{bc} = \Gamma^{a}_{bc} \mp H^{a}_{bc}$

$$\nabla^{+} J^{+} = \nabla^{-} J^{-} = 0 \,. \tag{6.17}$$

This establishes the T-duality at the level of the σ-models. Conformal invariance introduces further constraints. The relevant fact is that for both type of actions it is possible. For models written in twisted chiral superfields, there must be dilaton background if the target space is to be non-flat, while for models written in terms of chiral superfields, the target space can be any hyper-Kähler manifold without the need of a nontrivial dilaton background. They characterize the known examples of curved spaces for $N = 2$ α and β strings to propagate in. It would be interesting to investigate these geometrical approaches further.

7. Boundary conditions and D-branes

In the previous sections we studied in detail the two families of $N = 2$ strings α and β. A natural question to ask is for the classification of boundary conditions for the $N = 2$ string and their geometrical interpretation. These boundary conditions define D-branes of the $N = 2$ string theories. In this section we will discuss the subject briefly deferring the detailed analysis to [12, 13].[18]

A boundary breaks half of the superconformal symmetry. We require the gauge symmetry to be preserved by the boundary condition. The preserved half is a linear combination of the left and right chiral currents L^m:

$$L^m = U^m{}_n \tilde{L}^n \Big|_{\partial\Sigma} \,. \tag{7.1}$$

Here L^m correspond to the different generators of the chiral algebra. The allowed gluing coefficients $U^m{}_n$ are constrained by the algebra. Since the left and right chiral currents have the same algebra, consistent gluing condition must be in one-to-one correspondence with the automorphism group of the algebra. However, inner automorphism can be reduced to the identity by the symmetry of theory. They do not lead to inequivalent boundary conditions. Therefore we only have to consider the outer automorphisms.

One nontrivial outer automorphisms of $N = 2$ SCA is the $\mathbb{Z}_2$ conjugation automorphism

$$T \to T \,, \qquad J \to -J \,, \qquad G^{\pm} \to G^{\mp} \,. \tag{7.2}$$

In [26], this $\mathbb{Z}_2$ was used to classify the boundary conditions into two types: A-type and B-type. For the A-type boundary condition (in the closed string notation):

$$(L_n - \tilde{L}_n)\,|\,\rangle\rangle_A = (J_n - \tilde{J}_n)\,|\,\rangle\rangle_A = (G^{+}_n - \imath\tilde{G}^{-}_n)\,|\,\rangle\rangle_A = (G^{-}_n - \imath\tilde{G}^{+}_n)\,|\,\rangle\rangle_A = 0 \,, \tag{7.3}$$

[18] $N = 2$ boundary conditions for non-linear σ-models have been discussed recently in [25].

where $|\rangle\rangle_A$ denotes an A-type boundary state. For the B-type boundary condition:

$$(L_n - \tilde{L}_n)\,|\rangle\rangle_B = (J_n + \tilde{J}_n)\,|\rangle\rangle_B = (G_n^+ - \imath\tilde{G}_n^+)\,|\rangle\rangle_B = (G_n^- - \imath\tilde{G}_n^-)\,|\rangle\rangle_B = 0\,. \qquad (7.4)$$

Recall that in [26] one arrives at these boundary conditions by first requiring that a linear combination of T and $\tilde{T}$ be conserved *separately*, as well as linear combination of G and $\tilde{G}$, where $G = G^+ + G^-$. However, in $N = 2$ string theory there is no significance for this particular linear combination of supercharges, since it can be rotated to others by SO(2) rotations. Also for the $N = 2$ string theory T and J are on exactly the same footing, and arbitrary linear combinations thereof, are gauge symmetries. Therefore one can look for more general boundary conditions that mix the two. In [13] we discuss this in detail and arrive at other boundary conditions.

In flat space the σ-model is free and the boundary conditions can be solved exactly. We use the free field representation of the $N = (2, 2)$ algebra. Consider for instance the B boundary states. Equations eq. (7.4) can be solved by

$$(a^i_m - R^i{}_j\tilde{a}^j_{-m})\,|\rangle\rangle_B = (a^{\bar{i}}_m - R^{\bar{i}}{}_{\bar{j}}\tilde{a}^{\bar{j}}_{-m})\,|\rangle\rangle_B = 0\,, \qquad (7.5)$$

and

$$(\Psi^i_m - \imath R^i{}_j\tilde{\Psi}^j_{-m})\,|\rangle\rangle_B = (\Psi^{\bar{i}}_m - \imath R^{\bar{i}}{}_{\bar{j}}\tilde{\Psi}^{\bar{j}}_{-m})\,|\rangle\rangle_B = 0\,, \qquad (7.6)$$

where a^i_m and Ψ^i_m are the worldsheet bosonic and fermionic oscillators. The matrix R satisfies

$$(R^i{}_j)^* = R^{\bar{i}}{}_{\bar{j}}\,. \qquad (7.7)$$

For $(4, 0)$ signature R is an U(2) matrix, while for a $(2, 2)$ signature it is a U(1, 1) matrix. Note that equations eq. (7.5) determine the geometry and the (open string) gauge field of the D-brane, as

$$(\partial X^i - R^i{}_j\bar{\partial}X^j)\,|\rangle\rangle_B = (\partial X^{\bar{i}} - R^{\bar{i}}{}_{\bar{j}}\bar{\partial}X^{\bar{j}})\,|\rangle\rangle_B = 0\,. \qquad (7.8)$$

In the absence of gauge fields, vectors of R with eigenvalues (-1) and $(+1)$ correspond to directions normal and tangential to the D-brane, respectively. Examples of boundary state that satisfies the above conditions can be easily constructed [13]. The type A works in a similar way. One way is to perform a mirror symmetry on the right movers. Instead of $R^i{}_j$ and $R^{\bar{i}}{}_{\bar{j}}$, one would have $R^i_{\bar{j}}$ and $R^{\bar{i}}{}_j$, which again must be unitary or pseudo-unitary.

In [12] we will discuss the geometrical meaning of the boundary conditions when the target space is curved. The boundary conditions correspond to even-dimensional D-branes for the type β family and odd-dimensional for type α. They are mapped to each other under T-duality.

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A. Summaries of notations

A.1 Symbols and indices

- $\mathbb{R}^4$ vector: $I, J \ldots \in \{0, 1, 2, 3\}$.
- C^2 complex coordinates: $i, j \ldots ; \bar{i}\bar{j} \ldots \in \{1, 2\}$
- Affine SU(2) and SL(2, $\mathbb{R}$) currents: $i, j, \ldots \in \{1, 2, 3, \}$
- (Bi-)spinor, for SU(2) and SL(2, $\mathbb{R}$): $\alpha, \beta, \ldots \in \{1, 2\}$.

A.2 Worldsheet geometry

- Σ: Worldsheet manifold.
- ∂, $\tilde{\partial}$: Spacetime derivatives for the left movers and right movers respectively.
- ∂, $\bar{\partial}$: Worldsheet derivatives for the left and right movers respectively.
- $D(\tilde{D})$, $\bar{D}(\bar{\tilde{D}})$: Left (right) holomorphic/anti-holomorphic superderivatives in $N = 2$ superspace.

A.3 Worldsheet matter fields

- X^I: bosonic fields mapping worldsheet into target space.
- Ψ^I: fermionic fields mapping worldsheet into target space tangent bundle.
- $\mathbf{X}^I$: worldsheet superfields. $\mathbf{X}^I = X^I + \theta\Psi^I$.

A.3.1 Affine currents

- T: stress tensor (as a local field); L_m: Virasoro operator (as mode operator)
- $J(\tilde{J})$: Left (right) affine U(1) current of $N = 2$ SCA.
- $G^{\pm}(\tilde{G}^{\pm})$: Left (right) affine supercurrents of $N = 2$ SCA
- $J^i(\tilde{J}^i)$: Left (right) affine SU(2) or SL(2, $\mathbb{R}$) currents
- $G^{\alpha,\beta}$ $(\tilde{G}^{\alpha,\beta})$: Left (right) affine supercurrents of $N = 4$ SCA

A.3.2 Target space geometric data

- $\mathfrak{X}$: target space manifold
- $\mathcal{G}$: metric
- $\mathcal{J}$: complex structure. $\mathcal{J}\mathcal{J} = -1$.
- $\mathcal{K}$: Kähler form. $\mathcal{K} = \mathcal{G}\mathcal{J}$.
- $\mathcal{B}$: Rank 2 Anti-symmetric tensor
- $\mathcal{H}$: The field strength for $\mathcal{B}$: $\mathcal{H} = d\mathcal{B}$.
- D: covariant derivative
- Γ: affine connection
- Γ_0: Levi-Cevita connection based on $\mathcal{G}$.

A.3.3 Algebraic Invariants

- ϵ^{ijk}: $su(2) = so(3)$ structure constants.
- δ^{ij}: $su(2) = so(3)$ invariant Kronecker δ tensor.
- $(\sigma^i)^\alpha{}_\beta$: $su(2)$ invariant Pauli matrices.
- ε^{ij}_k: $sl(2,\mathbb{R}) = so(2,1)$ structure constants.
- η^{ij}: $sl(2,\mathbb{R}) = so(2,1)$ Killing metric.

A.3.4 Geometric objects

- S^2: 2-sphere, $\{(a_1,a_2,a_3)|a_1^2 + a_2^2 + a_3^2 = 1\}$.
- RP^2: real projective sphere, $S^2/(\vec{a} \sim -\vec{a})$.
- S_1^2: hyperboloid of two sheets, $\{(a_1,a_2,a_3)|a_1^2 - a_2^2 + a_3^2 = -1\}$.
- S_{1+}^2: Upper sheet of hyperboloid of two sheets, $s_1^2 \cap \{(a_1,a_2,a_3)|a_2 > 0\}$.

A.4 Conventions

$$\epsilon^{123} = 1 \tag{A.1}$$

$$\epsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \epsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{A.2}$$

$$\begin{aligned} (\sigma^1)^\alpha{}_\beta &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ (\sigma^2)^\alpha{}_\beta &= \begin{pmatrix} 0 & -\imath \\ \imath & 0 \end{pmatrix}, \\ (\sigma^3)^\alpha{}_\beta &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \tag{A.3}$$

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$$\sigma^{\pm} = \sigma^1 \pm i\sigma^2 . \tag{A.4}$$

$$(\sigma^i)_{\alpha\beta} = \epsilon_{\alpha\gamma}(\sigma^i)^{\gamma}{}_{\beta} , \qquad (\sigma^i)^{\alpha\beta} = (\sigma^i)^{\alpha}{}_{\gamma}\epsilon^{\gamma\beta} . \tag{A.5}$$

$$\eta^{ij} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{A.6}$$

$$\varepsilon_3^{12} = \varepsilon_1^{23} = \varepsilon_2^{13} = 1 . \tag{A.7}$$

B. Hyperboloid

Hyperboloid of two sheets is the described by the solution in $\mathbb{R}^3$ of the equation

$$x^2 + y^2 - z^2 = -1 . \tag{B.1}$$

It is so named because, obviously, it has two components (sheets), corresponding respectively to positive and negative values of z. It is known as the pseudo-2-sphere of index 1 with imaginary radius: S_1^2 [27]. The upper sheet, or equivalently the $\mathbb{Z}_2$ quotient by $z \to -z$, of S_1^2 is denoted by S_{1+}^2 in this paper.

One effective way to parameterize S_{1+}^2 is to use x and y. Hence there is a one-to-one map between R^2 and S_{1+}^2. Thus the polar coordinates θ and r of $\mathbb{R}^2$ is equally good for S_{1+}^2.

$\mathrm{O}(2,1)$ acts on S_1^2 just naturally as $\mathrm{O}(3)$ acts on S^2. The connected subgroup of $\mathrm{O}(2,1)$ is $\mathrm{SO}_0(2,1)$. It can be generated by a rotation of of the xy-plane with angle ϕ around the z axis, which acts by

$$r \to r, \qquad \theta \to \theta + \phi , \tag{B.2}$$

and a Lorentz boost along the x-direction with z as the "time." It acts transitively on plane sections of S_1^2 with fixed value of y.

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